

# THREE GENERALIZATIONS OF DAVENPORT–SCHINZEL SEQUENCES\*

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**Abstract.** We present new, and mostly sharp, bounds on the maximum length of certain generalizations of Davenport–Schinzel sequences. Among the results are sharp bounds on order- $s$  *double DS* sequences, for all  $s$ , sharp bounds on (double) *formation-free* sequences and new lower bounds on sequences avoiding *zig-zagging* patterns.

**Key words.** Extremal functions, forbidden subsequences, Davenport–Schinzel sequences.

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**1. Introduction.** A generalized Davenport–Schinzel (DS) sequence is one over a finite alphabet, say  $[n] = \{1, \dots, n\}$ , none of whose subsequences are isomorphic to a fixed forbidden sequence  $\sigma$  or a set of such sequences. (A sparsity criterion is also included in order to prohibit degenerate infinite sequences such as  $aaaaa \dots$ .) When  $\sigma$  is the alternating sequence  $abab \dots$  with length  $s + 2$  this definition reverts to that of standard order- $s$  DS sequences. Whereas standard DS sequences have countless applications in discrete and computational geometry, generalized DS sequences have found fewer applications [3, 6, 18, 21, 25, 29]. Whereas bounding the length of DS sequences is now essentially a closed problem [2, 16, 17], the most basic questions about generalized DS sequences are open, or have received only partial answers.

We are mainly interested in answering two questions about forbidden sequences. A purely quantitative question is to determine the maximum length  $\text{Ex}(\sigma, n)$  of a  $\sigma$ -free sequence over an  $n$ -letter alphabet, for specific  $\sigma$  or large classes of  $\sigma$ . An equally interesting question, particularly when  $\text{Ex}(\sigma, n)$  is superlinear in  $n$ , is to characterize the *structure* of  $\sigma$ -free sequences. There are infinitely many forbidden sequences one could study, but some classes of subsequences are more interesting than others, either because of their applications, or their intrinsic structure, or for historical reasons. In this article we focus on forbidden sequences that generalize, in various ways, the idea of an alternating sequence. In order to properly explain our results, in Section 1.4, we need to introduce some notation and terminology and to review the history of DS sequences and their generalizations, in Sections 1.1–1.3. For the moment we can take a high-level tour of the results. Following convention, let  $\lambda_s(n) = \text{Ex}(abab \dots, n)$  be the extremal function for order- $s$  DS sequences, where the alternating pattern has length  $s + 2$ .

*Double DS sequences.* The most modest way to generalize an alternating sequence  $abab \dots$  is simply to *double* each letter, transforming it to  $abbaabb \dots$ .<sup>1</sup> Double DS sequences were the first generalized DS sequences to be studied [1, 5, 14]. Let  $\lambda_s^{\text{dbl}}$  be the extremal function of order- $s$  double DS sequences. Davenport and Schinzel [5]

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<sup>1</sup>It is straightforward to show that repeating letters more than twice, or repeating the first and last at all, can affect the extremal function by at most a constant factor. See [1].

noted that  $\lambda_1^{\text{dbl}}(n)$  is linear (see [13, p. 13]) and Adamec, Klazar, and Valtr [1] proved that  $\lambda_2^{\text{dbl}}(n)$  is also linear, matching  $\lambda_1$  and  $\lambda_2$  up to constant factors. (The forbidden sequences here are *abba* and *abbaab*.) Klazar and Valtr [14] claimed without proof that  $\lambda_3^{\text{dbl}}(n) = \Theta(n\alpha(n))$ , which would match  $\lambda_3$  asymptotically [9]. However, this claim was later retracted [13]. Here  $\alpha(n)$  is the inverse-Ackermann function. We prove that  $\lambda_3^{\text{dbl}}(n)$  is, in fact,  $\Theta(n\alpha(n))$ , and more generally, that  $\lambda_s^{\text{dbl}}$  and  $\lambda_s$  are asymptotically equivalent for every order  $s$ .

*Formation-free Sequences.* Take any  $s + 1$  permutations over  $\{a, b\}$ . Regardless of one's choice, the concatenation of these permutations necessarily contains an alternating subsequence of length  $s + 2$ : the first permutation contributes two symbols and every subsequent permutation at least one. More generally, an  $(r, s + 1)$ -formation is obtained by concatenating  $s + 1$  permutations over an  $r$ -letter alphabet. Define  $\text{Form}(r, s + 1)$  to be the set of all  $(r, s + 1)$ -formations, and let  $\Lambda_{r,s}$  be the extremal function of  $\text{Form}(r, s + 1)$ -free sequences. The argument above shows that order- $s$  DS sequences are  $\text{Form}(2, s + 1)$ -free, which implies that  $\lambda_s(n) \leq \Lambda_{2,s}(n)$ . Klazar [10] introduced  $\text{Form}(r, s + 1)$ -free sequences as a ‘‘universal’’ method for finding upper bounds on  $\text{Ex}(\sigma, n)$ . If there exist  $r, s$  (and there always do) such that  $\sigma$  is contained in every member of  $\text{Form}(r, s + 1)$ , then  $\text{Ex}(\sigma, n) = O(\Lambda_{r,s}(n))$ .

A natural hypothesis, given [16, 17], is that  $\lambda_s$  and  $\Lambda_{r,s}$  are asymptotically equivalent, for all  $r$ . We prove that this hypothesis is false, which is quite surprising. One upshot of [2, 16, 17] is that when  $s \geq 7$  is odd,  $\lambda_s(n)$  and  $\lambda_{s-1}(n)$  are essentially indistinguishable, and that  $\lambda_5(n)$  and  $\lambda_4(n)$  are asymptotically distinguishable, but very similar. In contrast, we prove that, in general,  $\Lambda_{r,s}(n)$  behaves very differently at odd and even  $s$ . The extremal functions  $\lambda_s$  and  $\Lambda_{r,s}$  are asymptotically equivalent *only* when  $s \leq 3$ , or  $s \geq 4$  is even, or  $r = 2$ .

Just as DS sequences can be generalized to double DS sequences,  $\text{Form}(r, s + 1)$  can be transformed into a set  $\text{dblForm}(r, s + 1)$  by ‘‘doubling’’ it. Let  $\Lambda_{r,s}^{\text{dbl}}(n)$  be the extremal function of  $\text{dblForm}(r, s + 1)$ -free sequences. The function  $\Lambda_{r,s}^{\text{dbl}}$  was studied in a different, but essentially equivalent form by Cibulka and Kynčl [3]. We prove that  $\Lambda_{r,s}^{\text{dbl}}$  is asymptotically equivalent to  $\Lambda_{r,s}$  for all  $r, s$ . This fact is not surprising, but what *is* surprising is how many new techniques are needed to prove it when  $s = 3$ .

*Zig-zagging Patterns.* One way to view the alternating sequence *abab*... with length  $s + 2$  is as a *zig-zagging* pattern with  $s + 1$  *zigs* and *zags*. Generalized to larger alphabets, we obtain the  $N$ -shaped sequences, of the form *ab*...*zy*...*ab*...*z*, when  $s = 2$ , the  $M$ -shaped sequences *ab*...*zy*...*ab*...*zy*...*a*, when  $s = 3$ , the  $NW$ -shaped sequences *ab*...*zy*...*ab*...*zy*...*ab*...*z*, when  $s = 4$ , and so on. Klazar and Valtr [14] (see also [21]) proved that the extremal function of each  $N$ -shaped forbidden sequence is linear, matching  $\lambda_2(n)$ . See Valtr [29] for an application of  $N$ -shaped sequences to bounding the size of geometric graphs and Pettie [21] for an application of  $M$ -shaped sequences to bounding the complexity of the union of fat triangles.

Given [14, 21], one is tempted to guess that the extremal function for a zig-zagging forbidden sequence is, if not asymptotically equivalent to the corresponding order- $s$  DS sequence, at least close to it. We give lower bounds showing that for each  $t$ , there is an  $M$ -shaped forbidden sequence with extremal function  $\Omega(n\alpha^t(n))$  and an  $NW$ -shaped forbidden sequence with extremal function  $\Omega(n \cdot 2^{(1+o(1))\alpha^t(n)/t!})$ . Put a different way, in terms of their extremal functions  $M$ -shaped sequences may be similar to *ababa* but  $NW$ -shaped sequences bear no resemblance to *ababab*.

Our results on zig-zagging patterns are the least conclusive, and therefore offer the most opportunities for future research. They are based on a general, parameterized

method for constructing non-linear sequences.

**1.1. Sequence Notation and Terminology.** Let  $|\sigma|$  be the length of a sequence  $\sigma = (\sigma_i)_{1 \leq i \leq |\sigma|}$  and let  $\|\sigma\|$  be the size of its alphabet  $\Sigma(\sigma) = \{\sigma_i\}$ . Two equal length sequences are *isomorphic* if they are the same up to a renaming of their alphabets. We say  $\sigma$  is a *subsequence* of  $\sigma'$  if  $\sigma$  can be obtained by deleting symbols from  $\sigma'$ . The predicate  $\sigma < \sigma'$  asserts that  $\sigma$  is isomorphic to a subsequence of  $\sigma'$ . If  $\sigma \not< \sigma'$  we say  $\sigma'$  is  $\sigma$ -free. If  $P$  is a set of sequences,  $\sigma < P$  holds if  $\sigma < \sigma'$  for every  $\sigma' \in P$  and  $P \not< \sigma$  holds if  $\sigma' \not< \sigma$  for every  $\sigma' \in P$ . The alphabet size of  $P$  is  $\|P\| = \max_{\sigma \in P} \|\sigma\|$ . The assertion that  $\sigma$  *appears in* or *occurs in* or *is contained in*  $\sigma'$  means  $\sigma < \sigma'$ . The *projection* of a sequence  $\sigma$  onto  $G \subseteq \Sigma(\sigma)$  is obtained by deleting all non- $G$  symbols from  $\sigma$ . A sequence  $\sigma$  is  $k$ -sparse if whenever  $\sigma_i = \sigma_j$  and  $i \neq j$ , then  $|i - j| \geq k$ . A *block* is a sequence of distinct symbols. If  $\sigma$  is understood to be partitioned into a sequence of blocks,  $\llbracket \sigma \rrbracket$  is the number of blocks. The predicate  $\llbracket \sigma \rrbracket = m$  asserts that  $\sigma$  can be partitioned into at most  $m$  blocks. The extremal functions for *generalized* Davenport–Schinzel sequences are defined to be

$$\begin{aligned} \text{Ex}(\sigma, n, m) &= \max\{|S| : \sigma \not< S, \|S\| = n, \text{ and } \llbracket S \rrbracket \leq m\} \\ \text{Ex}(\sigma, n) &= \max\{|S| : \sigma \not< S, \|S\| = n, \text{ and } S \text{ is } \|\sigma\|\text{-sparse}\} \end{aligned}$$

where  $\sigma$  may be a single sequence or a set of sequences. The conditions “ $\llbracket S \rrbracket \leq m$ ” and “ $S$  is  $\|\sigma\|$ -sparse” guarantee that the extremal functions are finite. Note that  $\text{Ex}(\sigma, n, m)$  has no sparseness criterion. The extremal functions for order- $s$  DS sequences are defined to be

$$\lambda_s(n) = \text{Ex}(\overbrace{abab \cdots}^{\text{length } s+2}, n) \quad \text{and} \quad \lambda_s(n, m) = \text{Ex}(\overbrace{abab \cdots}^{\text{length } s+2}, n, m).$$

Since  $\|abab \cdots\| = 2$ , the sparseness criterion forbids only immediate repetitions.

**1.2. Davenport, Schinzel, Ackermann, Tarjan.** Davenport and Schinzel [4] observed that  $\lambda_1(n) = n$  and  $\lambda_2(n) = 2n - 1$ . It took several decades for all the other orders to be understood. The following theorem synthesizes results of Hart and Sharir [9], Agarwal, Sharir, and Shor [2], Klazar [12], Nivasch [16], and Pettie [17].

**THEOREM 1.1.** *Let  $\lambda_s(n)$  be the maximum length of a repetition-free sequence over an  $n$ -letter alphabet avoiding subsequences isomorphic to  $abab \cdots$  (length  $s + 2$ ). Then  $\lambda_s$  satisfies:*

$$\lambda_s(n) = \begin{cases} n & s = 1 \\ 2n - 1 & s = 2 \\ 2n\alpha(n) + O(n) & s = 3 \\ \Theta(n2^{\alpha(n)}) & s = 4 \\ \Theta(n\alpha(n)2^{\alpha(n)}) & s = 5 \\ n \cdot 2^{\alpha^t(n)/t!} + O(\alpha^{t-1}(n)) & s \geq 6, t = \lfloor \frac{s-2}{2} \rfloor. \end{cases}$$

Here  $\alpha(n)$  is the functional inverse of Ackermann's function discovered by Tarjan [28], defined as follows.

$$\begin{aligned} a_{1,j} &= 2^j & j \geq 1 \\ a_{i,1} &= 2 & i \geq 2 \\ a_{i,j} &= w \cdot a_{i-1,w} & i, j \geq 2 \\ & \text{where } w = a_{i,j-1} \end{aligned}$$

One may check that in the table  $(a_{i,j})$ , the first column is constant and the second column merely exponential:  $a_{i,1} = 2$  and  $a_{i,2} = 2^i$ . Ackermann-type growth only appears at the third column, motivating the following definition of the inverse functions.

$$\begin{aligned} \alpha(n, m) &= \min\{i \mid a_{i,j} \geq m, \text{ where } j = \max\{\lceil n/m \rceil, 3\}\} \\ \alpha(n) &= \alpha(n, n) \end{aligned}$$

There are numerous variants of Ackermann's function in the literature, all of which are equivalent inasmuch as their inverses differ by at most a constant. Observe that Theorem 1.1 is robust to perturbations of  $\alpha(n)$  by  $O(1)$ , so it does not depend on any particular definition of Ackermann's function or its inverse.<sup>2</sup>

**1.3. Generalizations of DS Sequences.** Certain classes of forbidden sequences have received significant attention. We review three systems for generalizing (standard) DS sequences, then mention some miscellaneous results in the area.

*Double DS Sequences.* Let  $\text{dbl}(\sigma)$  be obtained from  $\sigma$  by doubling each letter except for the first and last, for example,  $\text{dbl}(abcabc) = abbccaabbc$ . The extremal functions for order- $s$  double DS sequences are  $\lambda_s^{\text{dbl}}(n) = \text{Ex}(\text{dbl}(abab \cdots), n)$  and  $\lambda_s^{\text{dbl}}(n, m) = \text{Ex}(\text{dbl}(abab \cdots), n, m)$ , where the alternating sequence has length  $s+2$ . It is known that  $\lambda_1^{\text{dbl}}(n)$  and  $\lambda_2^{\text{dbl}}(n)$  are linear, matching  $\lambda_1$  and  $\lambda_2$  asymptotically. See Davenport and Schinzel [5], Adamec, Klazar, and Valtr [1], and Klazar [11, 13, p. 13]. Pettie [20, 21] proved that  $\lambda_3^{\text{dbl}}(n) = O(n\alpha^2(n))$  and  $\text{Ex}(\{abbaabba, ababab\}, n) = \Theta(n\alpha(n))$ , and that for  $s \geq 4$ ,  $\lambda_s^{\text{dbl}}(n)$  matched what were the best upper bounds on  $\lambda_s(n)$  at the time [16], namely  $\lambda_s^{\text{dbl}}(n) < n \cdot 2^{\alpha^t(n)/t! + O(\alpha^{t-1}(n))}$ , for even  $s$ , and  $\lambda_s^{\text{dbl}}(n) < n \cdot 2^{\alpha^t(n)(\log(\alpha(n)) + O(1))/t!}$ , for odd  $s$ .

*Formation-free Sequences.* Recall that  $\text{Form}(r, s+1)$  is defined to be the set of sequences obtained by concatenating  $s+1$  permutations over an  $r$ -letter alphabet. For example,  $abcd \ cbad \ badc \in \text{Form}(4, 3)$ . Let  $\Lambda_{r,s}(n) = \text{Ex}(\text{Form}(r, s+1), n)$  to be the extremal function for  $\text{Form}(r, s+1)$ -free sequences, with  $\Lambda_{r,s}(n, m)$  defined analogously.<sup>3</sup> It is straightforward to show that if  $\sigma$  is contained in every member of  $\text{Form}(r, s+1)$  then

$$\text{Ex}(\sigma, n, m) \leq \Lambda_{r,s}(n, m) \quad \text{and} \quad \text{Ex}(\sigma, n) = O(\Lambda_{r,s}(n)).$$

Nivasch [16] proved that any  $\sigma$  is contained in every member of  $\text{Form}(\|\sigma\|, \|\sigma\| - \|\sigma\| + 1)$ . Very recently Geneson, Prasad, and Tidor [8] showed that it suffices to consider a subset  $\text{Bin}_{r,s+1} \subset \text{Form}(r, s+1)$  consisting of *binary* patterns, where each of the  $s+1$  permutations is either  $12 \cdots (r-1)r$  or  $r(r-1) \cdots 21$ . By repeated application of the

<sup>2</sup>See Pettie [17, Remark 1.1] for a discussion of this notion of "Ackermann-invariance."

<sup>3</sup>The " $s+1$ " here is chosen to highlight the parallels with order- $s$  DS sequences. Recall that every  $\sigma \in \text{Form}(2, s+1)$  contains an alternating sequence  $abab \cdots$  with length  $s+2$ , hence  $\lambda_s(n) \leq \Lambda_{2,s}(n)$ .

Erdős–Szekeres theorem, they showed that every member of  $\text{Form}(r', s + 1)$  contains a member of  $\text{Bin}_{r, s+1}$ , where  $r' = (r - 1)^{2^s} + 1$ . Consequently, if  $\sigma$  is contained in every member of  $\text{Bin}_{r, s+1}$  then  $\text{Ex}(\sigma, n) = O(\Lambda_{r', s}(n))$ .

Nivasch [16], improving [10], gave the following upper bounds on  $\Lambda_{r, s}$ , for any  $r \geq 2, s \geq 1$ , where  $t = \lfloor \frac{s-2}{2} \rfloor$ . The lower bounds follow from previous [9, 2] and subsequent [17] constructions of order- $s$  DS sequences.

$$\Lambda_{r, s}(n) = \begin{cases} \Theta(n) & s \leq 2 \\ \Theta(n\alpha(n)) & s = 3 \\ \Theta(n2^{\alpha(n)}) & s = 4 \\ \Omega(n\alpha(n)2^{\alpha(n)}) \text{ and } O(n2^{\alpha(n)(\log \alpha(n)+O(1))}) & s = 5 \\ n \cdot 2^{\alpha^t(n)/t! + O(\alpha^{t-1}(n))} & \text{even } s \geq 6 \\ \Omega\left(n \cdot 2^{\alpha^t(n)/t! + O(\alpha^{t-1}(n))}\right) \\ \quad \text{and } O\left(n \cdot 2^{\alpha^t(n)(\log \alpha(n)+O(1))/t!}\right) & \text{odd } s \geq 7 \end{cases}$$

Note that  $\Lambda_{r, s}$  matches the behavior of  $\lambda_s$  when  $s \leq 3$  or  $s$  is even.

Cibulka and Kynčl [3] studied a problem on 0-1 matrices that is essentially equivalent to the following generalization of formation-free sequences. Define  $\text{dblForm}(r, s + 1)$  to be the set of all sequences over  $[r] = \{1, \dots, r\}$  that can be written  $\sigma_1 \dots \sigma_{s+1}$ , where  $\sigma_1$  and  $\sigma_{s+1}$  are permutations of  $[r]$  and  $\sigma_2, \dots, \sigma_s$  are sequences containing two copies of each symbol in  $[r]$ . Define  $\Lambda_{r, s}^{\text{dbl}}(n)$  and  $\Lambda_{r, s}^{\text{dbl}}(n, m)$  to be the extremal functions of  $\text{dblForm}(r, s + 1)$ -free sequences. Cibulka and Kynčl only considered  $\Lambda_{r, s}^{\text{dbl}}(n, m)$ . For consistency we state the bounds on  $\Lambda_{r, s}^{\text{dbl}}(n)$  they *would* have obtained using the available reductions from  $r$ -sparse to blocked sequences [16].<sup>4</sup> For any  $r \geq 2, s \geq 1$ , and  $t = \lfloor \frac{s-2}{2} \rfloor$ ,

$$\Lambda_{r, s}^{\text{dbl}}(n) = \begin{cases} \Theta(n) & s = 1 \\ \Omega(n) \text{ and } O(n\alpha(n)) & s = 2 \\ \Omega(n\alpha(n)) \text{ and } O(n\alpha^2(n)) & s = 3 \\ \Omega(n2^{\alpha(n)}) \text{ and } O(n\alpha^2(n)2^{\alpha(n)}) & s = 4 \\ \Omega(n\alpha(n)2^{\alpha(n)}) \text{ and } O(n2^{\alpha(n)(\log \alpha(n)+O(1))}) & s = 5 \\ n \cdot 2^{\alpha^t(n)/t! + O(\alpha^{t-1}(n))} & \text{even } s \geq 6 \\ \Omega\left(n \cdot 2^{\alpha^t(n)/t! + O(\alpha^{t-1}(n))}\right) \\ \quad \text{and } O\left(n \cdot 2^{\alpha^t(n)(\log \alpha(n)+O(1))/t!}\right) & \text{odd } s \geq 7 \end{cases}$$

The definition of  $\text{dblForm}(r, s + 1)$  may at first seem unnatural. Surely  $\text{dbl}(\text{Form}(r, s + 1)) = \{\text{dbl}(\sigma) \mid \sigma \in \text{Form}(r, s + 1)\}$  would be a more useful way to “double” the set  $\text{Form}(r, s + 1)$ . For example, it is known that  $abcacbc < \text{Form}(4, 4)$ , and therefore that  $\text{dbl}(abcacbc) < \text{dbl}(\text{Form}(4, 4))$ , but we cannot immediately conclude, as we would

<sup>4</sup>The only notable case here is  $s = 4$ . Cibulka and Kynčl proved that  $\Lambda_{r, 1}^{\text{dbl}}(n, m) = O(n + m)$ ,  $\Lambda_{r, 2}^{\text{dbl}}(n, m) = O((n + m)\alpha(n, m))$  and  $\Lambda_{r, 4}^{\text{dbl}}(n, m) = O((n + m)\alpha(n, m)2^{\alpha(n, m)})$ , which imply, by [16, Lem. 5.7], that  $\Lambda_{r, 2}^{\text{dbl}}(n) = O(n\alpha(n))$  and  $\Lambda_{r, 4}^{\text{dbl}}(n) = O(n\alpha^2(n)2^{\alpha(n)})$ .

like, that  $\text{Ex}(\text{dbl}(abcacbc), n) \leq \Lambda_{4,3}^{\text{dbl}}(n)$ . It turns out that the maximum length of  $\text{dblForm}(r, s+1)$ -free sequences and  $\text{dbl}(\text{Form}(r, s+1))$ -free sequences are the same asymptotically. The proof of Lemma 1.2 appears in the appendix.

LEMMA 1.2. *The following bounds hold for any  $r \geq 2, s \geq 1$ .*

$$\begin{aligned} \text{Ex}(\text{dbl}(\text{Form}(r, s+1)), n, m) &\leq r \cdot \Lambda_{r,s}^{\text{dbl}}(n, m) + 2rn \\ \text{Ex}(\text{dbl}(\text{Form}(r, s+1)), n) &= O(\Lambda_{r,s}^{\text{dbl}}(n)). \end{aligned}$$

*Zig-zagging Patterns.* Klazar and Valtr [14] introduced the  $N$ -shaped zig-zagging patterns  $\{N_k\}$ , where

$$N_k = 1 \ 2 \ \dots \ (k+1) \ k \ \dots \ 1 \ 2 \ \dots \ (k+1).$$

Note that  $N_k$ -free sequences generalize order-2 DS sequences since  $N_1 = abab$ . (The vertical placement of the symbols in  $N_k$  carries no meaning. It is only intended to improve readability.) It was shown [14, 21] that  $\text{Ex}(\text{dbl}(N_k), n) = O(n)$ , which matches  $\lambda_2(n)$  asymptotically. Pettie [21] proved that  $\text{Ex}(\{M_k, ababab\}, n) = \Theta(n\alpha(n))$ , matching  $\lambda_3(n)$ , where  $M_k$  is the  $k$ th  $M$ -shaped sequence,

$$M_k = 1 \ 2 \ \dots \ (k+1) \ k \ \dots \ 1 \ 2 \ \dots \ (k+1) \ k \ \dots \ 1.$$

See [29, 25, 6, 21] for applications of  $N$ - and  $M$ -shaped sequences.

A different way to view even-length alternating patterns  $abab\dots$  with length  $s+2$  is as a sequence of  $(s+2)/2$  zigs, without corresponding zags. When generalized to an  $r$ -letter alphabet we get the sequence  $(12\dots r)^{(s+2)/2}$ , which is contained in every member of  $\text{Bin}_{r,s+1}$  since at least  $\lceil \frac{s+1}{2} \rceil$  of the constituent permutations must be identical. It follows from [2, 8, 16] that  $\text{Ex}((1\dots r)^{(s+2)/2}, n) = \Theta(\Lambda_{r',s}(n)) = n \cdot 2^{(1+o(1))\alpha^t(n)/t!}$ , where  $r' = (r-1)^{2^s} + 1$  and  $t = \lfloor \frac{s-2}{2} \rfloor$ .

*Other Forbidden Patterns.* Much of the research on generalized DS sequences [1, 13, 14, 19, 20, 21, 22] has focussed on delineating linear and non-linear forbidden sequences. A  $\sigma$  is *linear* if  $\text{Ex}(\sigma, n) = O(n)$ . It is known that  $ababa$  and  $abcacbc$  are the only 2-sparse minimally non-linear sequences over three letters [14, 20, 21]. There are only a few varieties of sequences known to be linear. We have already seen that doubled  $N$ -shaped sequences ( $\text{dbl}(N_k)$ ) are in this category. Pettie [21, 19] proved that  $abcbccac$  is linear, and showed that if  $\pi_1, \pi_2$  are two permutations on the same alphabet, then  $\pi_1 \text{dbl}(\pi_2)$  is linear. For example,  $\text{Ex}(abcdeacceeabbd, n) = O(n)$ . More linear sequences can be generated via Klazar and Valtr's [14] splicing operation. If  $\sigma = \sigma_1 a a \sigma_2$  and  $\sigma'$  are linear, where  $\Sigma(\sigma) \cap \Sigma(\sigma') = \emptyset$ , then  $\sigma_1 a \sigma' a \sigma_2$  is also linear.

Other research has focussed on identifying cofinal sets of forbidden sequences, with respect to the total order on extremal functions.<sup>5</sup> Klazar's general upper bounds [10] imply that standard DS sequences  $\{(ab)^k\}$  are cofinal. Pettie [20], answering a question of Klazar [13], proved that the set of  $ababa$ -free forbidden sequences is also cofinal. This fact is witnessed by the two-sided *comb*-shaped sequences  $\{D_k\}$ , which generalize  $D_1 = abacacbc$ . Here  $D_k$  is defined to be

$$D_k = 1 \ 2 \ 1 \ 3 \ 1 \ 4 \ 1 \ \dots \ 1 \ (k+2) \ 1 \ (k+2) \ 2 \ (k+2) \ 3 \ \dots \ (k+1) \ (k+2).$$

<sup>5</sup>A set  $\mathcal{A}$  of sequences is cofinal if, for any  $\sigma$ , there is a  $\sigma' \in \mathcal{A}$  such that  $\text{Ex}(\sigma, n) = O(\text{Ex}(\sigma', n))$ .

**1.4. New Results.** In prior work [17] we showed that  $\lambda_s$  behaves very similarly at the odd and even orders. In this paper we prove, quite unexpectedly, that  $\Lambda_{r,s}$  matches  $\lambda_s$  *only* when  $s \leq 3$ , or  $s \geq 4$  is even, or  $r = 2$ . When  $s \geq 5$  is odd and  $r \geq 3$ ,  $\Lambda_{r,s}$  and  $\lambda_s$  diverge. Moreover, we prove that  $\lambda_s$  and  $\lambda_s^{\text{dbl}}$  are essentially equivalent, and that  $\Lambda_{r,s}$  and  $\Lambda_{r,s}^{\text{dbl}}$  are essentially equivalent.

**THEOREM 1.3. (Omnibus Bounds)** *For all  $s \geq 1$  and  $r = 2$ ,  $\lambda_s$ ,  $\lambda_s^{\text{dbl}}$ ,  $\Lambda_{r,s}$ , and  $\Lambda_{r,s}^{\text{dbl}}$  are asymptotically equivalent, namely,*

$$\lambda_s(n), \lambda_s^{\text{dbl}}(n), \Lambda_{2,s}(n), \Lambda_{2,s}^{\text{dbl}}(n) = \begin{cases} \Theta(n) & s \leq 2 \\ \Theta(n\alpha(n)) & s = 3 \\ \Theta(n2^{\alpha(n)}) & s = 4 \\ \Theta(n\alpha(n)2^{\alpha(n)}) & s = 5 \\ n \cdot 2^{\alpha^t(n)/t!} + O(\alpha^{t-1}(n)) & s \geq 6, \text{ where } t = \lfloor \frac{s-2}{2} \rfloor. \end{cases}$$

However, the behavior of  $\Lambda_{r,s}$  and  $\Lambda_{r,s}^{\text{dbl}}$  changes when  $r \geq 3$ . In particular,

$$\Lambda_{r,s}(n), \Lambda_{r,s}^{\text{dbl}}(n) = \begin{cases} \Theta(n) & s \leq 2 \\ \Theta(n\alpha(n)) & s = 3 \\ \Theta(n2^{\alpha(n)}) & s = 4 \\ n \cdot 2^{\alpha^t(n)(\log \alpha(n) + O(1))/t!} & \text{odd } s \geq 5 \\ n \cdot 2^{\alpha^t(n)/t!} + O(\alpha^{t-1}(n)) & \text{even } s \geq 6. \end{cases}$$

The new parts of Theorem 1.3 not covered by previous work [2, 3, 9, 16, 17] are

- (i) upper bounds on  $\lambda_s^{\text{dbl}}$ , for  $s \geq 4$ , which also cover  $\Lambda_{2,s}^{\text{dbl}}$ ,
- (ii) lower bounds on  $\Lambda_{r,s}$  for  $r \geq 3$  and odd  $s \geq 5$ ,
- (iii) a linear upper bound on  $\Lambda_{r,2}^{\text{dbl}}$ ,
- (iv) an  $O(n2^{\alpha(n)})$  upper bound on  $\Lambda_{r,4}^{\text{dbl}}$ , and
- (v) an  $O(n\alpha(n))$  upper bound on  $\Lambda_{r,3}^{\text{dbl}}$ , which also covers  $\lambda_3^{\text{dbl}}$ .

For task (i) we generalize (and simplify) the recent analysis of [17] to work for double DS sequences. This analysis *only* achieves tight bounds for  $s \geq 4$ . For task (ii) we give a construction of sequences that are  $\text{Form}(3, s+1)$ -free (but necessarily not  $\text{Form}(2, s+1)$ -free) with length  $n \cdot 2^{\alpha^t(n)(\log \alpha(n) + O(1))/t!}$ . Task (iii) requires no proof. It follows from the linearity of  $\text{dbl}(N_k)$ -free sequences. For task (iv) we give a single analysis of  $\Lambda_{r,s}^{\text{dbl}}$  that is tight for all  $r \geq 3, s \geq 4$ , but not  $s = 3$ . Task (v) is far and away the most difficult to prove. It requires the development of techniques new to the analysis of generalized DS sequences.

*Zig-zagging Patterns.* Recall that the  $N$ - and  $M$ -shaped sequences  $\{N_k, M_k\}$  generalize  $abab = N_1$  and  $ababa = M_1$ . Define  $Z_k$  to be the corresponding generalization of  $ababab = Z_1$ , that is,

$$Z_k = 1 \ 2 \ \dots \ (k+1) \ k \ \dots \ 1 \ 2 \ \dots \ (k+1) \ k \ \dots \ 1 \ 2 \ \dots \ (k+1).$$

We give a flexible new way to construct (and succinctly encode) nonlinear sequences that subsumes nearly all prior constructions [2, 9, 15, 16, 17, 20, 22]. Using the

new constructions we are able to show that for any  $t$ , there exists a  $k$  such that  $\text{Ex}(M_k, n) = \Omega(n\alpha^t(n))$  and an  $l$  such that  $\text{Ex}(Z_l, n) = \Omega(n \cdot 2^{(1+o(1))\alpha^t(n)/t!})$ . The bounds on  $M_k$ -free sequences are perhaps not too surprising, but they demonstrate that the extremal function for a *set* of forbidden sequences can be different than any member. (Recall that  $\text{Ex}(\{M_k, ababab\}, n) = \Theta(n\alpha(n))$  for any  $k$  [21].) The new bounds on  $Z_l$  show definitively that, in general, zig-zagging sequences are not closely tied to the corresponding DS sequences. In fact, the set  $\{Z_l\}$  is cofinal among all forbidden sequences, the other known cofinal sets being  $\{(ab)^k\}$  and two-sided combs  $\{D_k\}$ . Our new sequence constructions also let us show that the one-sided combs  $\{C_k\}$  behave differently than  $C_1 = abcacbc$ , where

$$C_k = 1 \ 2 \ 3 \ \cdots \quad (k+2) \quad (k+2) \quad (k+2) \quad (k+2) \quad \cdots \quad (k+1) \quad (k+2).$$

We prove  $\text{Ex}(C_k, n) = \Omega(n\alpha^k(n))$ .

**1.5. Organization.** In Section 2 we present sharp lower bounds on  $\text{Form}(r, s + 1)$ -free sequences. In Section 3 we review a number of standard sequence transformations and review the linear upper bounds on  $\lambda_s, \lambda_s^{\text{dbl}}, \Lambda_{r,s}$ , and  $\Lambda_{r,s}^{\text{dbl}}$  when  $s \in \{1, 2\}$ . In Section 4 we establish sharp upper bounds on  $\Lambda_{r,s}^{\text{dbl}}$ -free sequences, for all  $s \geq 4$ . Section 5 reviews the *derivation tree* structure introduced in [17], which is used in Sections 6 and 7. In Section 6 we present sharp upper bounds on  $\Lambda_{r,3}^{\text{dbl}}$  (and  $\lambda_3^{\text{dbl}}$ ) and in Section 7 we give sharp upper bounds on  $\lambda_s^{\text{dbl}}$  for all  $s \geq 4$ . Section 8 is devoted to a new, generalized construction of nonlinear sequences. We prove that, under appropriate parameterization, they are  $M_k$ -free,  $Z_k$ -free, and  $C_k$ -free. Some open problems are discussed in Section 9.

## 2. Lower Bounds on Formation-free Sequences.

**2.1. Composition and Shuffling.** We consider sequences made up of blocks, each of which is designated *live* or *dead*. To distinguish the two we use parentheses to indicate live blocks and angular brackets for dead blocks. The number of live blocks in  $T$  is  $\langle T \rangle$  and the number of both types is  $\|T\|$ . Our sequences are constructed through *composition* and two types of *shuffling* operations. These operations were implicit in all constructions since Hart and Sharir [9] but were usually presented in an ad hoc manner.

*Composition.* A sequence  $T$  over the alphabet  $\{1, \dots, \|T\|\}$  is in *canonical form* if symbols are ordered according to their first appearance in  $T$ . All sequences encountered in our construction are assumed to be in canonical form. To *substitute*  $T$  for a block  $B = (a_1, \dots, a_{\|T\|})$  means to replace  $B$  with a copy of  $T(B)$  under the alphabet mapping  $k \mapsto a_k$ . If  $T_{\text{mid}}$  is a sequence with  $\|T_{\text{mid}}\| = j$  and  $T_{\text{top}}$  a sequence in which live blocks have length  $j$ ,  $T_{\text{sub}} = T_{\text{top}} \circ T_{\text{mid}}$  is obtained by substituting for each live block  $B$  in  $T_{\text{top}}$  a copy  $T_{\text{mid}}(B)$ . The live/dead status of a block in  $T_{\text{sub}}$  is inherited from its status in  $T_{\text{top}}$  or  $T_{\text{mid}}$ , hence  $\langle T_{\text{sub}} \rangle = \langle T_{\text{top}} \rangle \cdot \langle T_{\text{mid}} \rangle$  and  $\|T_{\text{sub}}\| = \|T_{\text{top}}\| + \langle T_{\text{top}} \rangle (\|T_{\text{mid}}\| - 1)$ . If each symbol appears in  $\mu_{\text{top}}$  live blocks and  $\nu_{\text{top}}$  dead blocks in  $T_{\text{top}}$ , and  $\mu_{\text{mid}}$  live blocks and  $\nu_{\text{mid}}$  dead blocks in  $T_{\text{mid}}$ , then the corresponding multiplicities in  $T_{\text{sub}}$  are  $\mu_{\text{top}} \cdot \mu_{\text{mid}}$  and  $\nu_{\text{top}} + \mu_{\text{top}} \cdot \nu_{\text{mid}}$ .

*Shuffling.* Let  $T_{\text{bot}} = (L_1) \langle D_1 \rangle (L_2) \langle D_2 \rangle \cdots (L_l) \langle D_l \rangle$  be a sequence with  $l$  live blocks  $L_1, \dots, L_l$  and  $T_{\text{sub}} = (L'_1) \langle D'_1 \rangle (L'_2) \langle D'_2 \rangle \cdots (L'_l) \langle D'_l \rangle$  be a sequence whose live blocks  $L'_1, \dots, L'_l$  have length precisely  $l = \langle T_{\text{bot}} \rangle$ . The  $D$ s here represents zero or more dead blocks appearing between live blocks. The *postshuffle*  $T_{\text{sh}} = T_{\text{sub}} \otimes T_{\text{bot}}$

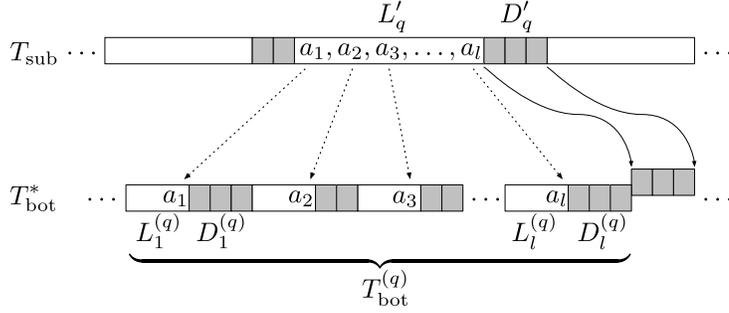


FIG. 1. Here  $L'_q = (a_1 \cdots a_l)$  is the  $q$ th live block of  $T_{\text{sub}}$  and  $T_{\text{bot}}^{(q)}$  is the  $q$ th copy of  $T_{\text{bot}}$  in  $T_{\text{bot}}^*$ . The sequence  $T_{\text{sub}} \otimes T_{\text{bot}}$  is obtained by shuffling  $L'_q$  into the live blocks of  $T_{\text{bot}}^{(q)}$  and inserting  $D'_q$  after  $T_{\text{bot}}^{(q)}$ .

is obtained by first forming the concatenation  $T_{\text{bot}}^*$  of  $l'$  copies of  $T_{\text{bot}}$ , each over an alphabet disjoint from the other copies and disjoint from  $\Sigma(T_{\text{sub}})$ . A copy of  $T_{\text{sub}}$  is shuffled into  $T_{\text{bot}}^*$  as follows. Let  $L'_q = (a_1 a_2 \cdots a_l)$  be the  $q$ th live block of  $T_{\text{sub}}$  and  $T_{\text{bot}}^{(q)} = (L_1^{(q)}) \langle D_1^{(q)} \rangle \cdots (L_l^{(q)}) \langle D_l^{(q)} \rangle$  be the  $q$ th copy of  $T_{\text{bot}}$  in  $T_{\text{bot}}^*$ . We substitute the following for  $T_{\text{bot}}^{(q)}$ , for all  $q$ , yielding  $T_{\text{sh}}$ .

$$(L_1^{(q)} a_1) \langle D_1^{(q)} \rangle \cdots (L_l^{(q)} a_l) \langle D_l^{(q)} D'_q \rangle$$

In other words, we insert  $a_p$  at the end of the  $p$ th live block in  $T_{\text{bot}}^{(q)}$  and insert all the dead blocks  $D'_q$  following  $L'_q$  in  $T_{\text{sub}}$  immediately after  $T_{\text{bot}}^{(q)}$ . See Figure 1. The *presuffle*  $T_{\text{sh}} = T_{\text{sub}} \otimes T_{\text{bot}}$  is formed in exactly the same way except that we insert  $a_p$  at the *beginning* of the block, that is, we substitute for  $T_{\text{bot}}^{(q)}$  the sequence  $(a_1 L_1^{(q)}) \langle D_1^{(q)} \rangle \cdots (a_l L_l^{(q)}) \langle D_l^{(q)} D'_q \rangle$ . In this section we consider only postshuffling whereas both pre- and postshuffling are used in Section 8.

**2.2. Construction of the Sequences.** Our  $\text{Form}(r, s + 1)$ -free sequences are constructed inductively, beginning with  $\text{Form}(r, 4)$ -free sequences  $\{T_\rho(i, j)\}_{i \geq 1, j \geq 0, \rho \geq 2}$ . Each  $T_\rho(i, j)$  consists of a mixture of live and dead blocks. The parameters  $i$  and  $j$  control the multiplicity of symbols and the length of live blocks, respectively. The length of dead blocks are guaranteed to be a multiple of  $\rho$ . Ignoring the role of  $\rho$ , this construction is essentially the same as the order-3 DS sequences presented in [9, 15, 30, 22].

$$\begin{aligned} T_\rho(1, j) &= (1 \cdots j) \langle 1 \cdots j \rangle && \text{one live block, one dead} \\ T_\rho(i, 0) &= ()^\rho && \rho \geq 2 \text{ empty live blocks, for } i \geq 2 \\ T_\rho(i, j) &= T_{\text{sub}} \otimes T_{\text{bot}} = (T_{\text{top}} \circ T_{\text{mid}}) \otimes T_{\text{bot}} \\ \text{where } T_{\text{bot}} &= T_\rho(i, j - 1) \\ T_{\text{mid}} &= (1 \cdots \lceil T_{\text{bot}} \rceil) \langle \lceil T_{\text{bot}} \rceil \cdots 1 \rangle && \text{one live block, one dead} \\ T_{\text{top}} &= T_\rho(i - 1, \lceil T_{\text{bot}} \rceil) \end{aligned}$$

Lemma 2.1 identifies some simple properties of  $T_\rho(i, j)$  that let us analyze its length and forbidden substructures.

LEMMA 2.1. *Let  $T = T_\rho(i, j)$  for some  $\rho \geq 2$ .*

1. *Live blocks of  $T$  consist solely of first occurrences and all first occurrences appear in live blocks.*
2. *Live blocks of  $T$  have length  $j$ .*
3. *All symbols appear  $i + 1$  times in  $T$ .*
4. *When  $i \geq 2$ , the number of live blocks and the length of dead blocks are both multiples of  $\rho$ .*
5. *As a consequence of Parts 1–3,  $|T| = (i + 1)\|T\| = (i + 1)j\langle T \rangle$ .*

*Proof.* All the claims trivially hold in the base cases, when  $i = 1$  or  $j = 0$ . Assume the claim holds inductively for pairs lexicographically smaller than  $(i, j)$ . Note that Part 1 holds for  $T_{\text{mid}}$ . If it holds for  $T_{\text{top}}$  and  $T_{\text{mid}}$  it clearly holds for  $T_{\text{sub}}$ , and if it holds for  $T_{\text{bot}}$  as well then it also holds for  $T_\rho(i, j) = T_{\text{sub}} \otimes T_{\text{bot}}$ .

Part 2 follows since, by the inductive hypothesis, live blocks in  $T_{\text{bot}} = T_\rho(i, j - 1)$  have length  $j - 1$  and exactly one symbol gets shuffled into each live block when forming  $T_\rho(i, j) = T_{\text{sub}} \otimes T_{\text{bot}}$ . Part 3 follows since the multiplicity of symbols in  $T_{\text{top}}$  is  $i$ , by the induction hypothesis, and the multiplicity in  $T_{\text{mid}}$  is 2, so the multiplicity of symbols in  $T_{\text{sub}}$  is  $i + 1$ . The multiplicity of symbols in  $T_{\text{bot}}$  is already  $i + 1$ , by the induction hypothesis, so all symbols occur in  $T$  with multiplicity  $i + 1$ .

Turning at last to Part 4, the claim is vacuous when  $i = 1$  and clearly holds when  $i \geq 2, j = 0$ . In general, if  $\langle T_{\text{bot}} \rangle = \langle T_\rho(i, j - 1) \rangle$  is a multiple of  $\rho$  then  $\langle T_\rho(i, j) \rangle$  is also a multiple of  $\rho$ . All dead blocks in  $T_\rho(i, j)$  are either (i) inherited from  $T_{\text{bot}}$ , or (ii) inherited from  $T_{\text{top}}$ , or (iii) are first introduced in  $T_{\text{sub}}$  as the second block in a copy of  $T_{\text{mid}} = (1 \cdots \langle T_{\text{bot}} \rangle) \langle \langle T_{\text{bot}} \rangle \cdots 1 \rangle$ . The inductive hypothesis implies that the length of category (i) blocks are multiples of  $\rho$ . When  $i \geq 3$  the inductive hypothesis also implies the length of category (ii) blocks are multiples of  $\rho$ . When  $i = 2$  we have  $T_{\text{top}} = T_\rho(1, \langle T_{\text{bot}} \rangle) = (1 \cdots \langle T_{\text{bot}} \rangle) \langle 1 \cdots \langle T_{\text{bot}} \rangle \rangle$ . By virtue of  $\langle T_{\text{bot}} \rangle$  being a multiple of  $\rho$ , the length of the lone dead block in  $T_{\text{top}}$  is a multiple of  $\rho$ . Category (iii) blocks satisfy the property for the same reason, since  $T_{\text{mid}} = (1 \cdots \langle T_{\text{bot}} \rangle) \langle \langle T_{\text{bot}} \rangle \cdots 1 \rangle$  and  $\langle T_{\text{bot}} \rangle$  is a multiple of  $\rho$ .  $\square$

LEMMA 2.2.  *$T_\rho(i, j)$  is an order-3 DS sequence, and hence Form( $r, 4$ )-free for all  $r \geq 2$ .*

*Proof.* The claim clearly holds in all base cases, so we can assume  $T = T_\rho(i, j)$  was formed from  $T_{\text{top}}, T_{\text{mid}}$ , and  $T_{\text{bot}}$ . Any occurrence of *ababa* could not have arisen from a shuffling event. If  $a \in \Sigma(T_{\text{top}})$  and  $b \in \Sigma(T_{\text{bot}}^*)$ , the projection of  $T$  onto  $\{a, b\}$  is  $|b^*ab^*|a^*$ , where the bars mark the boundary of  $b$ 's copy of  $T_{\text{bot}}$ . (The live block of  $T_{\text{sub}}$  shuffled into  $b$ 's  $T_{\text{bot}}$  contains the first occurrence of  $a$ . All other  $a$ s in  $T_{\text{sub}}$  are inserted after this copy of  $T_{\text{bot}}$ .) We could also not create an occurrence of *ababa* during a composition event, where  $a$  and  $b$  shared a live block in  $T_{\text{top}}$ . The projections of  $T_{\text{top}}$  and  $T_{\text{sub}}$  onto  $\{a, b\}$  would be, respectively, of the form  $(ab)a^*b^*$  and  $(ab)\langle ba \rangle a^*b^*$ , the latter being *ababa*-free.  $\square$

The  $U_s(i, j)$  sequences defined below have the property that all blocks are live and have length exactly  $j$  and all symbols occur  $\mu_{s,i}$  times, where the  $\mu$ -values are defined below. This contrasts with  $T_\rho(i, j)$ , where there is a mixture of live and dead blocks having non-uniform lengths. We define  $U_3(i, j)$  to be identical to  $T_j(i, j)$  as a sequence, but we interpret it as a sequence of live blocks of length exactly  $j$ . This is possible since, in  $T_j(i, j)$ , the length of live blocks is  $j$  and the length of each dead block is a multiple of  $j$ . Since all blocks in  $U_s$  are live we can use the identities  $\llbracket U_s(i, j) \rrbracket =$

$(\|U_s(i, j)\|)$  and  $|U_s(i, j)| = \mu_{s,i} \|U_s(i, j)\| = j \|U_s(i, j)\|$ . Sequences essentially the same as  $\{U_s\}$  were used in [20] to prove lower bounds on  $\text{Ex}(D_k, n)$ , where  $\{D_k\}$  are the two-sided combs defined in Section 1.3.

$$\begin{aligned}
U_2(i, j) &= (1 \cdots j)(j \cdots 1) && \text{two blocks, for all } i \\
U_s(i, 1) &= (1)^{\mu_{s,i}} && \mu_{s,i} \text{ identical blocks, for } i \geq 1, s \geq 3 \\
U_s(0, j) &= (1 \cdots j) && \text{one block, for } s \geq 3 \\
U_3(i, j) &= T_j(i, j) \text{ (reinterpreted)} && \text{for } i \geq 1, \text{ where } \rho = j \geq 2 \\
U_s(i, j) &= U_{\text{sub}} \otimes U_{\text{bot}} = (U_{\text{top}} \circ U_{\text{mid}}) \otimes U_{\text{bot}} \\
\text{where } U_{\text{bot}} &= U_s(i, j-1) \\
U_{\text{mid}} &= U_{s-2}(i, \|U_{\text{bot}}\|) \\
U_{\text{top}} &= U_s(i-1, \|U_{\text{mid}}\|)
\end{aligned}$$

The multiplicities  $\{\mu_{s,i}\}$  are defined as follows.

$$\begin{aligned}
\mu_{2,i} &= 2 && \text{for all } i \\
\mu_{3,i} &= i + 1 && \text{for all } i \\
\mu_{s,0} &= 1 && \text{for all } s \geq 4 \\
\mu_{s,i} &= \mu_{s,i-1} \mu_{s-2,i} && \text{for } s \geq 4 \text{ and } i \geq 1
\end{aligned}$$

LEMMA 2.3. *Let  $U = U_s(i, j)$ , where  $s \geq 2, i \geq 1, j \geq 1$ .*

1. *All symbols appear in  $U$  with multiplicity precisely  $\mu_{s,i}$ .*
2. *All blocks in  $U$  have length precisely  $j$ .*
3. *If  $a$  and  $b$  share a common block and  $a < b$  according to the canonical ordering of  $\Sigma(U)$ , then the projection of  $U$  onto  $\{a, b\}$  has the form either  $a^*b^*(ba)b^*a^*$  or  $a^*(ab)a^*b^*$ . Moreover, unless  $s = 2$ , every pair of symbols appear in at most one common block.*

*Proof.* Parts 1 and 2 hold in the base cases and follow easily by induction on  $s, i$ , and  $j$ . For Part 3, if  $b$  precedes  $a$  in their common block then, in some shuffling event,  $a \in \Sigma(U_{\text{sub}})$  was *postshuffled* into  $b$ 's copy of  $U_{\text{bot}}$  and all other copies of  $a$  were placed before or after this copy of  $U_{\text{bot}}$ , hence  $U$ 's projection onto  $\{a, b\}$  is  $a^*b^*(ba)b^*a^*$ . If  $a$  precedes  $b$  in their common block then this must be the *first* occurrence of  $b$  in  $U$  (otherwise  $b < a$  in the canonical ordering). By the same reasoning as above the projection of  $U$  onto  $\{a, b\}$  must be of the form  $a^*(ab)a^*b^*$ .  $\square$

In Lemma 2.4 we analyze the subsequences avoided by  $U_s$  and in Lemma 2.5 we lower bound the length of  $U_s$ .

LEMMA 2.4. *When  $s = 3$  or  $s \geq 2$  is even,  $U_s$  is an order- $s$  DS sequence and hence  $\text{Form}(2, s+1)$ -free. When  $s \geq 5$  is odd and  $r \geq 3$ ,  $U_s$  is  $\text{Form}(r, s+1)$ -free.*

*Proof.* The claim is clearly true for  $s = 2$  and Lemma 2.2 takes care of  $s = 3$ . Observe that  $ababab$  can never be introduced by a shuffling event. If  $a \in \Sigma(U_{\text{sub}})$  and  $b \in \Sigma(U_{\text{bot}}^*)$ , only one copy of  $a$  can appear between two  $b$ s; all others precede or follow  $b$ 's copy of  $U_{\text{bot}}$  in  $U_{\text{bot}}^*$ . Thus any alternating subsequence  $ab \cdots ab$  of length  $s+2 \geq 6$  must be introduced in  $U_{\text{sub}} = U_{\text{top}} \circ U_{\text{mid}}$  by composition. The projection of  $U_{\text{top}}$  onto  $\{a, b\}$  is of the form  $a^*b^*(ba)b^*a^*$ . Since  $U_{\text{mid}} = U_{s-2}(\cdot, \cdot)$  has order  $s-2$  and  $b$  precedes  $a$  in the canonical ordering of  $U_{\text{mid}}$ , its longest alternating subsequence

is  $bab \cdots ab$  (length  $s-1$ ), hence the longest alternating subsequence in  $U_{\text{sub}}$  has length  $s+1$ .

We now consider  $U = U_s(i, j)$ , where  $s \geq 5$  is odd. Define  $P_{s+1}$  to be the set of all sequences  $\sigma \in \{1, 2, 3\}^*$  such that  $\text{dbl}(\sigma)$  contains a subsequence of the form

$$123 \{123\}^{s-1} \{23\},$$

where  $\{123\}$  indicates *any* permutation of the symbols 1, 2, 3. We will prove by induction that  $U_s$  contains no  $P_{s+1}$  subsequence. This implies that it is  $\text{Form}(r, s+1)$ -free as well, for all  $r \geq 3$ . The claim holds at  $s = 3$  since all members of  $P_4$  contain 23232. For  $s \geq 5$ ,  $P_{s+1}$  could not have arisen from a shuffling event since every member of  $P_{s+1}$  contains a sequence isomorphic to  $ababab$ , for each pair of symbols  $\{a, b\} \subset \{1, 2, 3\}$ . It also could not have arisen from a composition event in which some *strict* subset of  $\{1, 2, 3\}$  appears in one block. Suppose a block in  $U_{\text{top}}$  contains 1 and 2 but not 3. The projection of  $U_{\text{top}}$  onto  $\{1, 2\}$  is of the form  $1^*2^*(21)2^*1^*$ . Even if there were 3s interspersed conveniently outside the block (21), substituting any  $U_{\text{mid}}$  for the block (21) could only create four permutations on  $\{123\}$  whereas we need at least  $s \geq 5$  such permutations.

We can therefore assume that any  $P_{s+1}$  sequence in  $U_s$ , say over the alphabet  $\{a, b, c\}$ , first arose in  $U_{\text{sub}}$  from a composition event in which some block  $B$  containing  $\{a, b, c\}$  is substituted for a copy  $U_{\text{mid}}(B)$ . By the inductive hypothesis  $U_{\text{mid}}$  is  $P_{s-1}$ -free. By Lemma 2.4, before the substitution the projection of  $U_{\text{top}}$  onto  $\{a, b, c\}$  is of the form

$$c^* b^* a^* (abc) a^* b^* c^*.$$

Some prefix of a  $P_{s+1}$  sequence is taken from  $c^*b^*a^*$ , some suffix of the  $P_{s+1}$  sequence is taken from  $a^*b^*c^*$ , and the remainder must come from the  $U_{\text{mid}}(B)$  substituted for  $(abc)$ . We consider three cases depending on the mapping from  $\{a, b, c\}$  to  $\{1, 2, 3\}$ .

*Case 1.* The mapping is  $c = 1$  and  $\{a, b\} = \{2, 3\}$ . The suffix of  $P_{s+1}$  can include *most*  $a$   $ab$ , that is, the final permutation  $\{23\}$  and the last letter of the last permutation of  $\{123\}$  if it is an  $a$ . The prefix of  $P_{s+1}$  can include at most  $cb$   $aa = 1233$ . Of course, since  $U_{\text{mid}}(B)$  is in canonical form and  $a < b < c$  according to the canonical order of  $\Sigma(U_{\text{mid}}(B))$ , we know  $a$  is the first letter among  $\{a, b, c\}$  to appear in  $U_{\text{mid}}(B)$ .<sup>6</sup> Thus, for  $P_{s+1}$  to appear in  $U_{\text{sub}}$  we would need  $U_{\text{mid}}(B)$  to contain  $abc\{abc\}^{s-3}\{bc\}$ , contradicting the  $P_{s-1}$ -freeness of  $U_{\text{mid}}(B)$ .

*Case 2.* The mapping is  $a = 1$  and  $\{b, c\} = \{2, 3\}$ . In this case the suffix can include at most  $ab$   $bc$ , but only if the last permutation on  $\{abc\}$  in  $P_{s+1}$  is exactly  $cab$ . Since  $U_{\text{mid}}(B)$  is in canonical form the first occurrence of  $a$  precedes those of  $b$  and  $c$ , so no useful prefix of  $P_{s+1}$  is provided by the  $c^*b^*a^*$  preceding  $B$  in  $U_{\text{top}}$ . For a  $P_{s+1}$  to appear in  $U_{\text{sub}}$  we would need  $U_{\text{mid}}(B)$  to contain  $abc\{abc\}^{s-2}c$ , contradicting the  $P_{s-1}$ -freeness of  $U_{\text{mid}}(B)$ .

*Case 3.* The mapping is  $b = 1$  and  $\{a, c\} = \{2, 3\}$ . The suffix can include at most  $a$   $ac$ . The prefix can apparently include as much as  $ba$ , but just as in Case 2, this is not a useful prefix. Since  $U_{\text{mid}}(B)$  is in canonical form the first  $c$  is already guaranteed to be preceded by  $ab$ . Thus, for  $U_{\text{sub}}$  to contain  $P_{s+1}$  we would need  $U_{\text{mid}}(B)$  to contain  $abc\{abc\}^{s-2}\{bc\}$ , contradicting the  $P_{s-1}$ -freeness of  $U_{\text{mid}}(B)$ .  $\square$

<sup>6</sup>Note that the canonical ordering of  $\{a, b, c\}$  within  $U_{\text{top}}$  and  $U_{\text{mid}}(B)$  are unrelated, and in fact typically the reversal of each other. If  $B$  contains neither the first occurrence of  $b$  nor  $c$  then  $c < b < a$  according to the canonical order of  $\Sigma(U_{\text{top}})$  but  $a < b < c$  according to the canonical order of  $\Sigma(U_{\text{mid}}(B))$ .

We have established that  $U_s$  is  $\text{Form}(r, s+1)$ -free and now need to lower bound its length.

LEMMA 2.5. *Fix  $s$  and let  $t = \lfloor (s-2)/2 \rfloor$ .*

1. *For even  $s$ ,  $\mu_{s,i} = 2^{\binom{i+t-1}{t}} = 2^{i^t/t!} + O(i^{t-1})$ .*

2. *For odd  $s$ ,  $\mu_{s,i} = \prod_{l=0}^i (i+1-l)^{\binom{l+t-1}{t-1}} = 2^{i^t(\log i)/t!} + O(i^t)$ .*

*Proof.* Consider the even case first. When  $i = 0$  we have  $\mu_{s,0} = 1 = 2^{\binom{0+t-1}{t}}$  and when  $s = 2, t = 0$  we have  $\mu_{2,i} = 2^{\binom{i+0-1}{0}} = 2$ . The claim holds for all even  $s \geq 4$  since, by Pascal's identity,  $\mu_{s,i} = \mu_{s,i-1} \cdot \mu_{s-2,i} = 2^{\binom{(i-1)+t-1}{t} + \binom{i+(t-1)-1}{t}} = 2^{\binom{i+t-1}{t}}$ . Clearly  $2^{\binom{i+t-1}{t}} \geq 2^{i^t/t!}$ .

For odd  $s$  the base case  $i = 0$  is trivial. When  $s = 5, t = 1$  we have  $\mu_{5,i} = \mu_{3,i} \mu_{3,i-1} \cdots \mu_{3,0} = (i+1)!$ , which can be expressed as  $\prod_{l=0}^i (i+1-l)^{\binom{l+t-1}{t-1}}$  since  $t = 1$  and  $\binom{l+0}{0} = 1$  for all  $l$ . For odd  $s \geq 7$  the bound follows by induction.

$$\begin{aligned} \mu_{s,i} &= \mu_{s,i-1} \cdot \mu_{s-2,i} \\ &= \prod_{l=0}^{i-1} ((i-1) + 1 - l)^{\binom{l+t-1}{t-1}} \cdot \prod_{l'=0}^i (i+1-l')^{\binom{l'+t-2}{t-2}} \\ &= \prod_{l''=0}^i (i+1-l'')^{\binom{l''+t-2}{t-1}} \cdot \prod_{l'=0}^i (i+1-l')^{\binom{l'+t-2}{t-2}} \\ &\quad \{l'' \stackrel{\text{def}}{=} l+1. \text{ When } l'' = 0, (i+1)^{\binom{t-2}{t-1}} = 1.\} \\ &= \prod_{l=0}^i (i+1-l)^{\binom{l+t-2}{t-1} + \binom{l+t-2}{t-2}} \\ &= \prod_{l=0}^i (i+1-l)^{\binom{l+t-1}{t-1}}. \end{aligned}$$

When  $s$  is odd, it is simpler to obtain asymptotic bounds on  $\log_2(\mu_{s,i})$  directly, without analyzing the closed-form expression above. Assuming inductively that  $\log_2(\mu_{s-2,i}) = i^{t-1}(\log i)/(t-1)! + O(i^{t-2})$ , where the constant hidden in the second term depends on  $s-2$ , we have

$$\begin{aligned} \log_2(\mu_{s,i}) &= \log_2(\mu_{s-2,i}) + \log_2(\mu_{s,i-1}) = \sum_{x=1}^i \log_2(\mu_{s-2,x}) \\ &= \sum_{x=1}^i \left[ \frac{x^{t-1} \log x}{(t-1)!} + O(x^{t-2}) \right] \\ &= \frac{i^t \log i}{t!} + O(x^{t-1}). \end{aligned}$$

Note that the sum is faithfully approximated by the integral  $\int_0^i x^{t-1}(\log x)/(t-1)! + O(x^{t-2}) dx = i^t(\log i)/t! + O(i^{t-1})$  as the two differ by  $O(i^{t-1})$ .  $\square$

It is a tedious exercise to show that for  $n = \|U_s(i, j)\|$  and  $m = \llbracket U_s(i, j) \rrbracket$ ,  $i = \alpha(n, m) + O(1)$  and  $j = \alpha(n) + O(1)$  when  $j = O(1)$ . (See [16, 20] for several examples of such calculations.) Lemmas 2.2, 2.4, and 2.5 establish all the lower bounds of Theorem 1.3, with the exception of  $\lambda_5(n) = \Omega(n\alpha(n)2^{\alpha(n)})$ , which is proved in [17].

REMARK 2.6. *It should be possible to improve the lower bounds on  $\Lambda_{3,s}$ , for odd  $s \geq 5$ , by substituting Nivasch's construction of order-3 DS sequences [16, §6] for*

$T_j(i, j)$  in the definition of  $U_3(i, j)$ . Nivasch's sequences are roughly twice as long as  $T_j(i, j)$ , which would lead to a  $2^{\binom{i+O(1)}{i}}$  factor improvement in  $\mu_{s,i}$ , for odd  $s \geq 5$ . The only technical issue is to deal with non-uniform block lengths. In the [16] construction there is no straightforward way to force dead blocks to have lengths that are multiples of some  $\rho$ . As a consequence, the block lengths in  $U_s(i, j)$  would also be non-uniform, but upper bounded by  $j$ .

**3. Sequence Transformations and Decompositions.** This section reviews some basic results and notation that is used throughout the article, sometimes without direct reference.

**3.1. Sparse Versus Blocked Sequences.** An  $m$ -block sequence can easily be converted to an  $r$ -sparse one by removing up to  $r - 1$  symbols in each block, except the first. This shows, for example, that  $\lambda_s(n, m) \leq \lambda_s(n) + m - 1$  and  $\Lambda_{r,s}^{\text{dbl}}(n, m) \leq \Lambda_{r,s}^{\text{dbl}}(n) + (r - 1)(m - 1)$ . However, converting an  $r$ -sparse sequence into one with  $O(n)$  blocks is, in general, not known to be possible without suffering some asymptotic loss. The following lemma generalizes reductions of Sharir [24] and Pettie [17] to  $\lambda_s^{\text{dbl}}, \Lambda_{r,s}$ , and  $\Lambda_{r,s}^{\text{dbl}}$ . In the interest of completeness we include a proof in Appendix A.

**LEMMA 3.1.** (Cf. Sharir [24], Füredi and Hajnal [7], and Pettie [17].) Define  $\gamma_s, \gamma_s^{\text{dbl}}, \gamma_{r,s}, \gamma_{r,s}^{\text{dbl}} : \mathbb{N} \rightarrow \mathbb{N}$  to be non-decreasing functions bounding the leading factors of  $\lambda_s(n), \lambda_s^{\text{dbl}}(n), \Lambda_{r,s}(n)$ , and  $\Lambda_{r,s}^{\text{dbl}}(n)$ , e.g.,  $\Lambda_{r,s}^{\text{dbl}} \leq \gamma_{r,s}^{\text{dbl}}(n) \cdot n$ . The following bounds hold.

$$\begin{aligned} \lambda_s(n) &\leq \gamma_{s-2}(n) \cdot \lambda_s(n, 2n) \\ \lambda_s^{\text{dbl}}(n) &\leq (\gamma_{s-2}^{\text{dbl}}(n) + 4) \cdot \lambda_s^{\text{dbl}}(n, 2n) \\ \lambda_s(n) &\leq \gamma_{s-2}(\gamma_s(n)) \cdot \lambda_s(n, 3n) \\ \lambda_s^{\text{dbl}}(n) &\leq (\gamma_{s-2}^{\text{dbl}}(\gamma_s^{\text{dbl}}(n)) + 4) \cdot \lambda_s^{\text{dbl}}(n, 3n) \\ \Lambda_{r,s}(n) &\leq \gamma_{r,s-2}(n) \cdot \Lambda_{r,s}(n, 2n) + 2n \\ \Lambda_{r,s}^{\text{dbl}}(n) &\leq (\gamma_{r,s-2}^{\text{dbl}}(n) + O(1)) \cdot \Lambda_{r,s}^{\text{dbl}}(n, 2n) \\ \Lambda_{r,s}(n) &\leq \gamma_{r,s-2}(\gamma_{r,s}(n)) \cdot \Lambda_{r,s}(n, 3n) + 2n \\ \Lambda_{r,s}^{\text{dbl}}(n) &\leq (\gamma_{r,s-2}^{\text{dbl}}(\gamma_{r,s}^{\text{dbl}}(n)) + O(1)) \cdot \Lambda_{r,s}^{\text{dbl}}(n, 3n), \end{aligned}$$

where the  $O(1)$  terms depend on  $r$  and  $s$ .

**3.2. Reductions Between Formation-free Sequences and DS Sequences.**

It is not immediate from the definitions that the extremal functions  $\lambda_s, \lambda_s^{\text{dbl}}, \Lambda_{2,s}$ , and  $\Lambda_{2,s}^{\text{dbl}}$  are closely related. Lemma 3.2 is used to reduce the number of facts that must be established to prove Theorem 1.3: lower bounds on  $\lambda_s$  apply to the other extremal functions and upper bounds on  $\Lambda_{2,s}^{\text{dbl}}$  apply to the other extremal functions.

**LEMMA 3.2.** *The following inequalities hold for all  $s$ .*

$$\begin{aligned} \lambda_s(n) &\leq \Lambda_{2,s}(n) &\leq \lambda_s^{\text{dbl}}(n) &\leq \Lambda_{2,s}^{\text{dbl}}(n) + 2n &\leq 5 \cdot \lambda_s^{\text{dbl}}(n) + 2n \\ \lambda_s(n, m) &\leq \Lambda_{2,s}(n, m) &\leq \lambda_s^{\text{dbl}}(n, m) &\leq \Lambda_{2,s}^{\text{dbl}}(n, m) + n &\leq 3 \cdot \lambda_s^{\text{dbl}}(n, m) + n. \end{aligned}$$

Refer to Appendix A for proof of Lemma 3.2.

**3.3. Linearity at Orders 1 and 2.** We bound the length of sequences inductively through the use of recurrences. The induction bottoms out when  $s \in \{1, 2\}$ , so we need to handle these two orders directly. Lemma 3.3 summarizes linear bounds on  $\lambda_s, \lambda_s^{\text{dbl}}, \Lambda_{r,s}$ , and  $\Lambda_{r,s}^{\text{dbl}}$  that were discovered by Davenport and Schinzel [4, 5],

Klazar [10, 13], Klazar and Valtr [14], Füredi and Hajnal [7], and Pettie [21]. A proof of Lemma 3.3 appears in Appendix A.

LEMMA 3.3. *At orders  $s = 1$  and  $s = 2$ , the extremal functions  $\lambda_s, \lambda_s^{\text{dbl}}, \Lambda_{r,s}$ , and  $\Lambda_{r,s}^{\text{dbl}}$  obey the following.*

$$\begin{array}{lll}
\lambda_1(n) = n & \lambda_1(n, m) = n + m - 1 & \\
\lambda_2(n) = 2n - 1 & \lambda_2(n, m) = 2n + m - 2 & [4] \\
\lambda_1^{\text{dbl}}(n) = 3n - 2 & \lambda_1^{\text{dbl}}(n, m) = 2n + m - 2 & [5, 13] \\
\lambda_2^{\text{dbl}}(n) < 8n & \lambda_2^{\text{dbl}}(n, m) < 5n + m & [11, 7] \\
\Lambda_{r,1}(n) = \Lambda_{r,1}^{\text{dbl}}(n) < rn & \Lambda_{r,1}(n, m) = \Lambda_{r,1}^{\text{dbl}}(n, m) < n + (r - 1)m & [10] \\
\Lambda_{r,2}(n) < 2rn & \Lambda_{r,2}(n, m) < 2n + (r - 1)m & [10] \\
\Lambda_{r,2}^{\text{dbl}}(n) < 6^r rn & \Lambda_{r,2}^{\text{dbl}}(n, m) < 2 \cdot 6^{r-1}(n + m/3) & [21]
\end{array}$$

The linear bound on  $\Lambda_{r,2}^{\text{dbl}}$  is a consequence of bounds on  $\text{dbl}(N_{r-1})$ -free sequences [14, 21], though this connection was not noted earlier [3].

**3.4. Sequence Decomposition.** We adopt and extend the sequence decomposition notation from [17]. This style of decomposition goes back to Hart and Sharir [9] and Agarwal, Sharir, and Shor [2], and has been used many times since then [3, 10, 16, 20]. This notation is used liberally throughout Sections 4–7.

Let  $S$  be a sequence over an  $n = \|S\|$  letter alphabet consisting of  $m = \llbracket S \rrbracket$  blocks. (It may be that  $S$  avoids some forbidden sequences, but this has no bearing on the decomposition.) A partition of  $S$  into  $\hat{m}$  intervals  $S_1 \cdots S_{\hat{m}}$  is called *uniform* if  $m_1 = \cdots = m_{\hat{m}-1}$  are equal powers of two and  $m_{\hat{m}}$  may be smaller, where  $m_q = \llbracket S_q \rrbracket$  is the number of blocks in the  $q$ th interval. A symbol is *global* if it appears in multiple intervals and *local* otherwise. Let  $\check{S} = \check{S}_1 \cdots \check{S}_{\hat{m}}$  and  $\hat{S} = \hat{S}_1 \cdots \hat{S}_{\hat{m}}$  be the projections of  $S$  onto local and global symbols, so  $|S| = |\check{S}| + |\hat{S}|$ . Define  $\hat{n} = \|\hat{S}\|$  to be the size of the global alphabet and  $\hat{n}_q = \|\hat{S}_q\|$  and  $\check{n}_q = \|\check{S}_q\|$  to be number of global and local symbols in  $\Sigma(S_q)$ , so  $n = \hat{n} + \sum_{1 \leq q \leq \hat{m}} \check{n}_q$ .

A global symbol  $a \in \Sigma(\hat{S}_q)$  is classified as *first*, *last*, or *middle* if no  $a$ s appear before  $S_q$ , no  $a$ s appear after  $S_q$ , or  $a$ s appear both before and after  $S_q$ .<sup>7</sup> Let  $\acute{S}_q, \grave{S}_q, \bar{S}_q < \hat{S}_q$  be the projections of  $\hat{S}_q$  onto symbols classified as first, last, and middle in  $\hat{S}_q$ ; let  $\acute{n}_q, \grave{n}_q$ , and  $\bar{n}_q$  be the sizes of the alphabets  $\Sigma(\acute{S}_q), \Sigma(\grave{S}_q)$ , and  $\Sigma(\bar{S}_q)$ . Define  $\acute{S}, \grave{S}$ , and  $\bar{S}$  to be subsequences of first, last, and middle occurrences, namely

$$\begin{aligned}
\acute{S} &= \acute{S}_1 \acute{S}_2 \cdots \acute{S}_{\hat{m}-1} \\
\grave{S} &= \grave{S}_2 \cdots \grave{S}_{\hat{m}-1} \grave{S}_{\hat{m}} \\
\bar{S} &= \bar{S}_2 \cdots \bar{S}_{\hat{m}-1}.
\end{aligned}$$

Note that  $\hat{S}_1 = \acute{S}_1$  consists solely of first occurrences and  $\hat{S}_{\hat{m}} = \grave{S}_{\hat{m}}$  consists solely of last occurrences, so  $\bar{S}$  is empty if  $\hat{m} = 2$ . These notational conventions will be applied to sequences and other objects defined later. For example, the diacritical marks  $\check{\cdot}, \hat{\cdot}, \acute{\cdot}, \grave{\cdot}$ , and  $\bar{\cdot}$  will be applied to objects pertaining to local, global, first, last, and middle symbols, respectively. Moreover, whenever we define a new subsequence of  $S_q$ , say  $\check{S}_q$ , quantities and objects pertaining to  $\check{S}_q$  will be indicated with the same diacritical mark, such as  $\check{n}_q = \|\check{S}_q\|$ .

<sup>7</sup>Note that if  $a \in \Sigma(\hat{S}_q)$  is classified as first, all of the possibly *many* occurrences of  $a$  in  $S_q$  are “first” occurrences.

The global contracted sequence  $\hat{S}' = B_1 \cdots B_{\hat{m}}$  is obtained by *contracting* each interval  $\hat{S}_q$  to a single block  $B_q$  consisting of some permutation of  $\Sigma(\hat{S}_q)$ . Unless specified otherwise, the symbols in  $B_q$  are ordered according to their *first* occurrence in  $\hat{S}_q$ . It follows that  $\hat{S}' < \hat{S}$ , so  $\hat{S}'$  inherits any forbidden sequences of  $\hat{S}$ .

**4. Upper Bounds on dblForm( $r, s$ )-free Sequences.** In this section we give recurrences for the extremal functions of Form( $r, s+1$ )-free sequences and dblForm( $r, s+1$ )-free sequences. Lemmas 4.4 and 4.5 give closed-form upper bounds on the length of such sequences in terms of Ackermann's function. These bounds on  $\Lambda_{r,s}$  and  $\Lambda_{r,s}^{\text{dbl}}$  are sharp, except for  $\Lambda_{2,s}$  and  $\Lambda_{2,s}^{\text{dbl}}$ , when  $s \geq 5$  is odd, and  $\Lambda_{r,3}^{\text{dbl}}$ , for any  $r \geq 2$ . These exceptions are addressed in Sections 6 and 7.

**4.1. A Recurrence for  $\Lambda_{r,s}$ .** In reading the proofs of Recurrences 4.1 and 4.3 one should keep in mind that all extremal functions are superadditive. For example,

$$\Lambda_{r,s}(n_1, m_1) + \Lambda_{r,s}(n_2, m_2) \leq \Lambda_{r,s}(n_1 + n_2, m_1 + m_2).$$

RECURRENCE 4.1. *Define  $n$  and  $m$  to be the alphabet size and block count parameters. For any  $\hat{m} \geq 2$ , any block partition  $\{m_q\}_{1 \leq q \leq \hat{m}}$ , and any alphabet partition  $\{\hat{n}\} \cup \{\check{n}_q\}_{1 \leq q \leq \hat{m}}$ ,  $\Lambda_{r,s}$  obeys the following recurrences, for any fixed  $r \geq 2, s \geq 3$ .*

When  $\hat{m} = 2$ ,

$$\Lambda_{r,s}(n, m) \leq \sum_{q \in \{1,2\}} \Lambda_{r,s}(\check{n}_q, m_q) + \Lambda_{r,s-1}(2\hat{n}, m)$$

and when  $\hat{m} > 2$ ,

$$\Lambda_{r,s}(n, m) \leq \sum_{q=1}^{\hat{m}} \Lambda_{r,s}(\check{n}_q, m_q) + 2 \cdot \Lambda_{r,s-1}(\hat{n}, m) + \Lambda_{r,s-2}(\Lambda_{r,s}(\hat{n}, \hat{m}) - 2\hat{n}, m).$$

*Proof.* We adopt the sequence decomposition notation from Section 3.4. The contribution of local symbols is  $\sum_q |\check{S}_q| \leq \sum_q \Lambda_{r,s}(\check{n}_q, m_q)$ . As each symbol in  $\check{S}_q$  appears at least once after  $S_q$ , each  $\check{S}_q$  is a Form( $r, s$ )-free sequence, it follows that

$$\sum_{q=1}^{\hat{m}-1} |\check{S}_q| \leq \sum_{q=1}^{\hat{m}-1} \Lambda_{r,s-1}(\check{n}_q, m_q) \leq \Lambda_{r,s-1} \left( \sum_{q=1}^{\hat{m}-1} \check{n}_q, \sum_{q=1}^{\hat{m}-1} m_q \right) = \Lambda_{r,s-1}(\hat{n}, m - m_{\hat{m}}).$$

A symmetric statement is true for each  $\hat{S}_q$ , hence the contribution of last occurrences is  $\sum_q |\hat{S}_q| \leq \Lambda_{r,s-1}(\hat{n}, m - m_1)$ . If  $\hat{m} = 2$  then we have accounted for all symbols, and by superadditivity  $\Lambda_{r,s-1}(\hat{n}, m_1) + \Lambda_{r,s-1}(\hat{n}, m_2) \leq \Lambda_{r,s-1}(2\hat{n}, m)$ .

If  $\hat{m} > 2$  then we must also count middle symbols. Each symbol in  $\bar{S}_q$  appears at least once before  $\hat{S}_q$  and at least once afterward. This implies that  $\bar{S}_q$  is Form( $r, s-1$ )-free, hence

$$\begin{aligned} \sum_q |\bar{S}_q| &\leq \sum_q \Lambda_{r,s-2}(\bar{n}_q, m_q) \\ &\leq \Lambda_{r,s-2} \left( \sum_q \bar{n}_q, \sum_q m_q \right) && \text{superadditivity} \\ &= \Lambda_{r,s-2}(|\hat{S}'| - 2\hat{n}, m - m_1 - m_{\hat{m}}) && (4.1) \\ &< \Lambda_{r,s-2}(\Lambda_{r,s}(\hat{n}, \hat{m}) - 2\hat{n}, m) && \hat{S}' \text{ is Form}(r, s+1)\text{-free} \end{aligned}$$

Equality (4.1) follows since  $\sum_q \bar{n}_q$  counts the number of middle occurrences of symbols in  $\hat{S}'$ , that is, the length of  $\hat{S}'$  less  $2\hat{n}$  for first and last occurrences.  $\square$

**4.2. A Recurrence for  $\Lambda_{r,s}^{\text{dbl}}$ .** Recall that  $\Lambda_{r,s}^{\text{dbl}}(n, m)$  was defined to be the extremal function for  $\text{dblForm}(r, s+1)$ -free,  $m$ -block sequences over an  $n$ -letter alphabet. Here  $\text{dblForm}(r, s+1)$  is the set of sequences over the alphabet  $[r] = \{1, \dots, r\}$  of the form  $\sigma_1 \cdots \sigma_{s+1}$ , where  $\sigma_1$  and  $\sigma_{s+1}$  contain one occurrence of each symbol in  $[r]$  and  $\sigma_2, \dots, \sigma_s$  contain exactly two occurrences of each symbol in  $[r]$ .

REMARK 4.2. *The definition of  $\Lambda_{r,s}^{\text{dbl}}$  has one annoying property. Suppose  $S$  is a sequence and  $S'$  a contracted version of it in which each occurrence of a symbol represents two or more occurrences in  $S$ . We would like to say that if  $S$  is  $\text{dblForm}(r, s+1)$ -free then  $S'$  is  $\text{Form}(r, s+1)$ -free, but this is not strictly true. For example, suppose  $S'$  contained the  $\text{Form}(2, 4)$  sequence  $ab|b(a|b)a|ab$ , where the bars separate the four constituent permutations over  $\{a, b\}$  and the parentheses mark the boundaries of one block  $B$  in  $S'$ . If we substitute  $aa$  and  $bb$  for all  $as$  and  $bs$  outside  $B$ , and substitute  $abab$  for  $B$ , we find that  $S$  may only contain  $aabb\ bb\ (abab)\ aa\ aabb$ , which contains no  $\text{dblForm}(2, 4)$  sequence. On the other hand, if occurrences in  $S'$  represent at least three occurrences in  $S$ , and symbols in the blocks of  $S'$  are sorted according to the 2nd occurrence in the corresponding subsequence of  $S$ , then  $S'$  is  $\text{Form}(r, s+1)$  free if  $S$  is  $\text{dblForm}(r, s+1)$ -free.*

We can easily “force” blocks in  $S'$  to represent at least three corresponding occurrences in the original sequence. Suppose we are given an initial  $\text{dblForm}(r, s+1)$ -free sequence  $S^*$ . Obtain  $S$  from  $S^*$  by retaining every other occurrence of each symbol, so  $S$  is also  $\text{dblForm}(r, s+1)$ -free and  $|S| \geq |S^*|/2$ . When bounding  $|S|$  inductively we may construct a contracted version  $S'$  whose occurrences represent at least two occurrences in  $S$ , and hence at least three occurrences in  $S^*$ . (One subtlety here is that  $S'$  will be a subsequence of  $S^*$ , not necessarily  $S$ , since we order symbols in the blocks of  $S'$  according to their position in  $S^*$ .)

In Recurrence 4.3 (and Recurrences 6.1 and 7.4 later on) we use the inference  $[S \text{ is } \text{dbl}(\sigma)\text{-free}] \rightarrow [S' \text{ is } \sigma\text{-free}]$ , knowing that the bounds we obtain on the given extremal function may be off by a factor of two.

RECURRENCE 4.3. *Define  $n$  and  $m$  to be the alphabet size and block count parameters. For any  $\hat{m} \geq 2$ , block partition  $\{m_q\}_{1 \leq q \leq \hat{m}}$ , and alphabet partition  $\{\hat{n}\} \cup \{\check{n}_q\}_{1 \leq q \leq \hat{m}}$ ,  $\Lambda_{r,s}^{\text{dbl}}$  obeys the following recurrences, for any fixed  $r \geq 2, s \geq 3$ .*

When  $\hat{m} = 2$ ,

$$\Lambda_{r,s}^{\text{dbl}}(n, m) \leq \sum_{q \in \{1, 2\}} \Lambda_{r,s}^{\text{dbl}}(\check{n}_q, m_q) + \Lambda_{r,s-1}^{\text{dbl}}(2\hat{n}, m) + 2\hat{n}$$

and when  $\hat{m} > 2$ ,

$$\begin{aligned} \Lambda_{r,s}^{\text{dbl}}(n, m) \leq & \sum_{q=1}^{\hat{m}} \Lambda_{r,s}^{\text{dbl}}(\check{n}_q, m_q) + \Lambda_{r,s}^{\text{dbl}}(\hat{n}, \hat{m}) + 2 \cdot \Lambda_{r,s-1}^{\text{dbl}}(\hat{n}, m) \\ & + \Lambda_{r,s-2}^{\text{dbl}}(\Lambda_{r,s}(\hat{n}, \hat{m}) - 2\hat{n}, m) + 2 \cdot \Lambda_{r,s}(\hat{n}, \hat{m}) \end{aligned}$$

*Proof.* We consider the case when  $\hat{m} > 2$  first. Let  $S$  be a  $\text{dblForm}(r, s+1)$ -free sequence. The contribution of local symbols is  $\sum_q |\check{S}_q| \leq \sum_q \Lambda_{r,s}^{\text{dbl}}(\check{n}_q, m_q)$ . If a global

symbol appears exactly once in some  $\hat{S}_q$  that occurrence is called a *singleton*. Let  $\dot{S}$  be the subsequence of  $\hat{S}$  consisting of singletons. Clearly  $\dot{S}$  can be partitioned into  $\hat{m}$  blocks, hence  $|\dot{S}| \leq \Lambda_{r,s}^{\text{dbl}}(\hat{n}, \hat{m})$ . Remove all singleton occurrences from  $\hat{S}$  and let  $\ddot{S}$  be what remains. Classify occurrences in  $\ddot{S}_q$  as *first*, *middle*, and *last* according to whether they do not occur before, do not occur after, or occur both before and after interval  $q$  in  $\hat{S}$  (not in  $\ddot{S}$ .) Let  $\acute{S}, \grave{S}, \bar{S} < \ddot{S}$  be the subsequences of first, last, and middle occurrences. Obtain  $\acute{S}_q^-$  (and  $\grave{S}_q^-$ ) from  $\acute{S}_q$  (and  $\grave{S}_q$ ) by removing the last (and first) occurrence of each symbol, and obtain  $\bar{S}_q^-$  from  $\bar{S}_q$  by removing both the first and last occurrence of each symbol. It follows that both  $\acute{S}_q^-$  and  $\grave{S}_q^-$  are  $\text{dblForm}(r, s)$ -free, and that  $\bar{S}_q^-$  is  $\text{dblForm}(r, s - 1)$ -free. The contribution of first and last non-singleton occurrences in  $\ddot{S}$  is therefore at most

$$\sum_q \left[ \Lambda_{r,s-1}^{\text{dbl}}(\acute{n}_q, m_q) + \acute{n}_q + \Lambda_{r,s-1}^{\text{dbl}}(\grave{n}_q, m_q) + \grave{n}_q \right] \leq 2 \cdot \left[ \Lambda_{r,s-1}^{\text{dbl}}(\hat{n}, m) + \hat{n} \right].$$

Form  $\ddot{S}'$  from  $\ddot{S}$  by contracting each interval into a single block. Since  $\ddot{S}$  is  $\text{dblForm}(r, s + 1)$ -free,  $\ddot{S}'$  must be  $\text{Form}(r, s + 1)$ . (See Remark 4.2.) Therefore, the contribution of middle non-singleton occurrences is at most

$$\begin{aligned} \sum_q \left[ \Lambda_{r,s-2}^{\text{dbl}}(\bar{n}_q, m_q) + 2\bar{n}_q \right] &\leq \Lambda_{r,s-2}^{\text{dbl}} \left( \sum_q \bar{n}_q, \sum_q m_q \right) + 2 \cdot \sum_q \bar{n}_q \\ &= \Lambda_{r,s-2}^{\text{dbl}}(|\ddot{S}'| - 2\hat{n}, m) + 2(|\ddot{S}'| - 2\hat{n}) \\ &\leq \Lambda_{r,s-2}^{\text{dbl}}(\Lambda_{r,s}(\hat{n}, \hat{m}) - 2\hat{n}, m) + 2 \cdot \Lambda_{r,s}(\hat{n}, \hat{m}) - 4\hat{n}. \end{aligned}$$

When  $\hat{m} = 2$  there are no middle occurrences and, in the worst case, no singletons. The total number of first and last occurrences is  $(\Lambda_{r,s-1}^{\text{dbl}}(\hat{n}, m_1) + \hat{n}) + (\Lambda_{r,s-1}^{\text{dbl}}(\hat{n}, m_2) + \hat{n}) \leq \Lambda_{r,s-1}^{\text{dbl}}(2\hat{n}, m) + 2\hat{n}$ . This concludes the proof of the recurrence.  $\square$

Lemma 4.4 gives explicit upper bounds on  $\Lambda_{r,s}$  and  $\Lambda_{r,s}^{\text{dbl}}$  in terms of inductively defined coefficients  $\{\pi_{s,i}, \pi_{s,i}^{\text{dbl}}\}$  and the  $i$ th row-inverse of Ackermann's function. One should keep in mind, when reading this lemma and similar lemmas, that we will ultimately substitute  $\alpha(n, m) + O(1)$  for  $i$ , and that this choice makes the dependence on the block count  $m$  negligible.

LEMMA 4.4. *Fix parameters  $i \geq 1$ ,  $r \geq 2$ ,  $s \geq 3$ , and  $c \geq s - 2$ . Let  $n, m$  be the alphabet size and block count and let  $j$  be minimal such that  $m \leq (a_{i,j})^c$ . Then  $\Lambda_{r,s}$  and  $\Lambda_{r,s}^{\text{dbl}}$  are bounded as follows.*

$$\begin{aligned} \Lambda_{r,s}(n, m) &\leq \pi_{s,i}(n + O((cj)^{s-2}m)) \\ \Lambda_{r,s}^{\text{dbl}}(n, m) &\leq \pi_{s,i}^{\text{dbl}}(n + O((cj)^{s-2}m)), \end{aligned}$$

where the asymptotic notation hides a constant depending only on  $r$ . The coefficients

$\{\pi_{s,i}, \pi_{s,i}^{\text{dbl}}\}$  are defined as follows.

$$\begin{aligned}
\pi_{1,i} &= \pi_{1,i}^{\text{dbl}} = 1 \\
\pi_{2,i} &= 2 \\
\pi_{s,1} &= 2\pi_{s-1,1} = 2^{s-1} \\
\pi_{s,i} &= 2\pi_{s-1,i} + \pi_{s-2,i}(\pi_{s,i-1} - 2) \\
\pi_{2,i}^{\text{dbl}} &= 2 \cdot 6^{r-1} \\
\pi_{s,1}^{\text{dbl}} &= 2\pi_{s-1,1}^{\text{dbl}} + 1 < (6^{r-1} + 1)2^s \\
\pi_{s,i}^{\text{dbl}} &= \pi_{s,i-1}^{\text{dbl}} + 2\pi_{s-1,i}^{\text{dbl}} + (\pi_{s-2,i}^{\text{dbl}} + 2)\pi_{s,i-1}
\end{aligned} \tag{4.2}$$

$$\pi_{s,i}^{\text{dbl}} = \pi_{s,i-1}^{\text{dbl}} + 2\pi_{s-1,i}^{\text{dbl}} + (\pi_{s-2,i}^{\text{dbl}} + 2)\pi_{s,i-1} \tag{4.3}$$

The proof is by induction over tuples  $(s, i, j)$ , where  $c$  and  $r$  are regarded as fixed. (The base cases when  $s \in \{1, 2\}$  follow from Lemma 3.3.) At the base case  $i = 1$  we let  $j$  be minimal such that  $m \leq a_{1,j}$ . By invoking Recurrence 4.1 with  $\hat{m} = 2$  is it easy to show that  $\Lambda_{r,s}(n, m) \leq \pi_{s,1}(n + O(j^{s-2}m))$ , where the constant hidden by the asymptotic notation does not depend on  $s$  or  $c$ . This also implies that  $\Lambda_{r,s}(n, m) \leq \pi_{s,1}(n + O((cj)^{s-2}m))$  when  $j$  is defined to be minimal such that  $m \leq a_{1,j}^c$ , since  $a_{1,j}^c = a_{1,cj} = 2^{cj}$ . In the general case, when  $i > 1$ , we apply Recurrence 4.1 using a uniform block partition with width  $w^c = a_{i,j-1}^c$ , so

$$\hat{m} = \lceil m/w^c \rceil \leq (a_{i,j})^c / (a_{i,j-1})^c = (a_{i-1,w})^c.$$

We invoke the inductive hypothesis with parameters  $i, j - 1$  on sequences with  $w^c$  blocks (namely  $\{\check{S}_q\}$ ). On sequences with  $m$  blocks (such as  $\acute{S}, \grave{S}$ ) we invoke the inductive hypothesis with  $i, j$  and on sequences with  $\hat{m}$  blocks we invoke it with  $i - 1, w$ . The induction goes through smoothly so long as the coefficients  $\{\pi_{s,i}, \pi_{s,i}^{\text{dbl}}\}$  are defined as in Lemma 4.4, Eqns. (4.2,4.3). See [17, Appendix B] for several examples of such proofs in this style.<sup>8</sup>

LEMMA 4.5. *The ensemble  $\{\pi_{s,i}, \pi_{s,i}^{\text{dbl}}\}_{s \geq 3, i \geq 1}$  satisfies the following, where  $t = \lfloor \frac{s-2}{2} \rfloor$ .*

$$\begin{aligned}
\pi_{3,i} &= 2i + 2 \\
\pi_{3,i}^{\text{dbl}} &= \Theta(i^2) \\
\pi_{4,i}, \pi_{4,i}^{\text{dbl}} &= \Theta(2^i) \\
\pi_{5,i}, \pi_{5,i}^{\text{dbl}} &\leq 2^i(i + O(1))! \\
\pi_{s,i}, \pi_{s,i}^{\text{dbl}} &\leq 2^{\binom{i+O(1)}{t}} && \text{for even } s > 4 \\
\pi_{s,i}, \pi_{s,i}^{\text{dbl}} &\leq 2^{\binom{i+O(1)}{t} \log(2(i+1)/e)} && \text{for odd } s > 5
\end{aligned}$$

<sup>8</sup>For an alternative approach see Nivasch [16, §3]. It differs in two respects. First, it refers to the slowly growing row-inverses of Ackermann's function rather than using the 'j' parameter of Ackermann's function. Second, there is no equivalent to our 'c' parameter in [16], which leads to a system of two recurrences, one for the leading factor of the  $n$  term, and one for the leading factor of the  $j^{s-2}m$  term. For yet another style of analysis, which leads to the same recurrences for  $\pi_{s,i}$  and  $\pi_{s,i}^{\text{dbl}}$ , see Nivasch [16, §4], Cibulka and Kynčl [3, §2], or Sundar [26].

*Proof.* First consider the case when  $s \in \{3, 4\}$ . Eqn. (4.2) simplifies to

$$\begin{aligned}\pi_{3,i} &= 2 + \pi_{3,i-1} \\ \pi_{4,i} &= 2\pi_{3,i} + 2(\pi_{4,i-1} - 2)\end{aligned}$$

One proves by induction that  $\pi_{3,i} = 2i + 2$  and  $\pi_{4,i} = 10 \cdot 2^i - 4(i + 2)$ . Using these identities, Eqn. (4.3) can be simplified to

$$\begin{aligned}\pi_{3,i}^{\text{dbl}} &= \pi_{3,i-1}^{\text{dbl}} + 2 \cdot (2 \cdot 6^{r-1}) + (1 + 2)(2i - 2) \\ \pi_{4,i}^{\text{dbl}} &\leq \pi_{4,i-1}^{\text{dbl}} + 2 \cdot \pi_{3,i}^{\text{dbl}} + (2 \cdot 6^{r-1} + 2)(10 \cdot 2^{i-1} - 4(i + 1)).\end{aligned}$$

A short proof by induction shows  $\pi_{3,i}^{\text{dbl}} \leq 6\binom{i+1}{2} + 4 \cdot 6^{r-1}(i + 1)$  and that  $\pi_{4,i}^{\text{dbl}} \leq 20(6^{r-1} + 2)2^i$ . In the general case we have, for  $s \geq 5$ ,

$$\begin{aligned}\pi_{s,i} &\leq 2\pi_{s-1,i} + \pi_{s-2,i}\pi_{s,i-1} \\ &= 2\pi_{s-1,i} + \pi_{s-2,i}(2\pi_{s-1,i-1} + \pi_{s-2,i-1}(2\pi_{s-1,i-2} \\ &\quad + \pi_{s-2,i-2}(\cdots + \pi_{s-2,2}\pi_{s,1}) \cdots)) \\ &= \sum_{l=0}^{i-2} 2\pi_{s-1,i-l} \cdot \prod_{k=0}^{l-1} \pi_{s-2,i-k} + \pi_{s,1} \cdot \prod_{k=0}^{i-2} \pi_{s-2,i-k}\end{aligned}\tag{4.4}$$

When  $s = 5$  we have  $\pi_{s-1,i} = \Theta(2^i)$  and  $\pi_{s-2,i} = 2(i + 1)$ , so (4.4) can be written

$$\begin{aligned}&= \sum_{l=0}^{i-2} \Theta(2^{i-l}) \cdot 2(i + 1)2i \cdots 2(i + 2 - l) + \pi_{s,1} \cdot 2(i + 1)2i2(i - 1) \cdots 2(3) \\ &= \Theta(2^i \cdot (i + 1)!) = 2^{(i+O(1)) \log(\frac{2(i+1)}{e})}\end{aligned}$$

We prove that there are constants  $\{C_s\}$  such that  $\pi_{s,i} \leq 2^{\binom{i+C_s}{t}}$  when  $s$  is even and  $\pi_{s,i} \leq 2^{\binom{i+C_s}{t} \log(2(i+1)/e)}$  when  $s$  is odd. The analysis above shows that  $C_4$  and  $C_5$  exist. When  $s > 4$  is even, (4.4) is bounded by

$$\leq \sum_{l=0}^{i-2} 2^{\binom{i-l+C_{s-1}}{t-1} \log(\frac{2(i-l+1)}{e})} \cdot \prod_{k=0}^{l-1} 2^{\binom{i-k+C_{s-2}}{t-1}} + \pi_{s,1} \cdot \prod_{k=0}^{i-2} 2^{\binom{i-k+C_{s-2}}{t-1}}\tag{4.5}$$

By Pascal's identity  $\sum_{k=0}^x \binom{i-k+C_{s-2}}{t-1} = \binom{i+1+C_{s-2}}{t} - \binom{i-x+C_{s-2}}{t}$ , so (4.5) is bounded by

$$\begin{aligned}&\leq 2^{\binom{i+1+C_{s-2}}{t}} \cdot \left( \sum_{l=0}^{i-2} 2^{\binom{i-l+C_{s-1}}{t-1} \log(\frac{2(i-l+1)}{e})} - \binom{i-l+1+C_{s-2}}{t} + \pi_{s,1} \right) \\ &\leq 2^{\binom{i+1+C_s}{t}}, \text{ for some sufficiently large } C_s.\end{aligned}\tag{4.6}$$

The sum in (4.6) clearly converges as  $i \rightarrow \infty$ , though for some constant values of  $i-l$  (depending on  $C_{s-1}$  and  $C_{s-2}$ ),  $\binom{i-l+C_{s-1}}{t-1} \log(2(i-l+1)/e)$  may be significantly larger than  $\binom{i-l+1+C_{s-2}}{t}$ . When  $s > 5$  is odd the calculations are similar. By the inductive hypothesis, (4.4) is bounded by

$$\begin{aligned} &\leq \sum_{l=0}^{i-2} 2^{\binom{i-l+C_{s-1}}{t}} \cdot \prod_{k=0}^{l-1} 2^{\binom{i-k+C_{s-2}}{t-1} \log(\frac{2(i-k+1)}{e})} + \pi_{s,1} \cdot \prod_{k=0}^{i-2} 2^{\binom{i-k+C_{s-2}}{t-1} \log(\frac{2(i-k+1)}{e})} \\ &\leq 2^{\binom{i+1+C_{s-2}}{t} \log(\frac{2(i+1)}{e})} \cdot \left( \sum_{l=0}^{i-2} 2^{\binom{i-l+C_{s-1}}{t} - \binom{i-l+1+C_{s-2}}{t} \log(\frac{2(i+1)}{e})} + \pi_{s,1} \right) \\ &\leq 2^{\binom{i+1+C_s}{t} \log(\frac{2(i+1)}{e})}, \text{ for some sufficiently large } C_s. \end{aligned} \quad (4.7)$$

Turning to  $\pi_{s,i}^{\text{dbl}}$ , we have

$$\begin{aligned} \pi_{s,i}^{\text{dbl}} &= \pi_{s,i-1}^{\text{dbl}} + 2\pi_{s-1,i}^{\text{dbl}} + (\pi_{s-2,i}^{\text{dbl}} + 2)\pi_{s,i-1} \\ &= \pi_{s,1}^{\text{dbl}} + \sum_{l=0}^{i-2} [2\pi_{s-1,i-l}^{\text{dbl}} + (\pi_{s-2,i-l}^{\text{dbl}} + 2)\pi_{s,i-1-l}] \end{aligned} \quad (4.8)$$

It is straightforward to show that when  $s \geq 4$ , the bounds on  $\pi_{s,i}$  also hold for  $\pi_{s,i}^{\text{dbl}}$  with respect to different constants  $\{D_s\}$ . When  $s = 5$ , Eqn. (4.8) becomes

$$\begin{aligned} \pi_{5,i}^{\text{dbl}} &= \pi_{5,1}^{\text{dbl}} + \sum_{l=0}^{i-2} (2 \cdot \Theta(2^{i-l}) + (\Theta(i-l)^2 + 2) \cdot \Theta(2^{i-1-l}(i-l)!)) \\ &= \Theta(2^i(i+2)!) \leq 2^{(i+D_5) \log(\frac{2(i+1)}{e})}, \text{ for a sufficiently large } D_5. \end{aligned}$$

When  $s > 4$  is even, Eqn. (4.8) implies, by the inductive hypothesis, that

$$\begin{aligned} \pi_{s,i}^{\text{dbl}} &\leq \pi_{s,1}^{\text{dbl}} + \sum_{l=0}^{i-2} \left[ 2^{\binom{i-l+D_{s-1}}{t-1} \log(\frac{2(i-l+1)}{e}) + 1} + (2^{\binom{i-l+D_{s-2}}{t-1} \log(\frac{2(i-l+1)}{e})} + 2) 2^{\binom{i-1-l+C_s}{t} \log(\frac{2(i-l)}{e})} \right] \\ &\leq 2^{\binom{i+1+D_s}{t} \log(\frac{2(i+1)}{e})}, \text{ for a sufficiently large } D_s. \end{aligned}$$

When  $s > 5$  is odd,

$$\begin{aligned} \pi_{s,i}^{\text{dbl}} &\leq \pi_{s,1}^{\text{dbl}} + \sum_{l=0}^{i-2} \left[ 2^{\binom{i-l+D_{s-1}}{t} \log(\frac{2(i-l+1)}{e}) + 1} + (2^{\binom{i-l+D_{s-2}}{t-1} \log(\frac{2(i-l+1)}{e})} + 2) 2^{\binom{i-1-l+C_s}{t} \log(\frac{2(i-l)}{e})} \right] \\ &\leq 2^{\binom{i+1+D_s}{t} \log(\frac{2(i+1)}{e})}, \text{ for a sufficiently large } D_s. \end{aligned}$$

□

Given that Lemma 4.5 holds for all  $i$ , one chooses  $i$  to be minimum such that the ‘ $m$ ’ term does not dominate, that is, the minimum  $i$  for which  $j \leq 3$  or  $(cj)^{s-2} \leq n/m$ . It is straightforward to show that  $i = \alpha(n, m) + O(1)$  is optimal, which immediately gives bounds on  $\Lambda_{r,s}(n, m)$  and  $\Lambda_{r,s}^{\text{dbl}}(n, m)$  analogous to those claimed for  $\Lambda_{r,s}(n)$  and  $\Lambda_{r,s}^{\text{dbl}}(n)$  in Theorem 1.3, excluding the case  $s = 3$ , which is dealt with in Section 6. In

order to obtain bounds on  $\Lambda_{r,s}(n)$  and  $\Lambda_{r,s}^{\text{dbl}}(n)$  we invoke Lemma 3.1. For example, it states that  $\Lambda_{r,s}(n) = \gamma_{r,s-2}(\gamma_{r,s}(n)) \cdot \Lambda_{r,s}(n, 3n) + 2n$ , where  $\gamma_{r,s}(n)$  is a non-decreasing upper bound on  $\Lambda_{r,s}(n)/n$ . The  $\gamma_{r,s-2}(\gamma_{r,s}(n))$  factor may not be constant, but it does not affect the error tolerance already in the bounds of Theorem 1.3.<sup>9</sup>

**REMARK 4.6.** *Our lower and upper bounds on  $\Lambda_{r,s}(n)$  are tight (when  $r \geq 3$ ) inasmuch as they are both of the form  $n \cdot 2^{\alpha^t(n)/t! + O(\alpha^{t-1}(n))}$  when  $s \geq 4$  is even and  $n \cdot 2^{\alpha^t(n)(\log \alpha(n) + O(1))/t!}$  when  $s \geq 5$  is odd. However, it is only when  $s$  is even that these bounds are sharp in the Ackermann-invariant sense of [17, Remark 1.1], that is, invariant under  $\pm O(1)$  perturbations in the definition of  $\alpha(n)$ . For example, our lower and upper bounds on  $\Lambda_{r,5}(n)$  are  $n \cdot (\alpha(n) + O(1))!$  and  $n \cdot 2^{\alpha(n)}(\alpha(n) + O(1))!$ . The  $2^{\alpha(n)}$  factor gap could probably be closed by substituting Nivasch's construction of order-3 DS sequences [16, §6] for  $U_3(i, j)$  in Section 2, which would lead to sharp, Ackermann-invariant bounds of  $\Lambda_{r,5}(n) = n \cdot 2^{\alpha(n)}(\alpha + O(1))!$ . With a more careful analysis of the recurrence for  $\pi_{s,i}$  it should be possible to obtain sharp, Ackermann-invariant bounds on  $\Lambda_{r,s}(n)$  for all odd  $s$ .*

**5. Derivation Trees.** Derivation trees were introduced in [17] to model hierarchical decompositions of sequences. They are instrumental in our analysis of  $\text{dblForm}(r, 4)$ -free sequences, in Section 6, and of double DS sequences, in Section 7. Throughout this section we use the sequence decomposition notation defined in Section 3.4.

A recursive decomposition of a sequence  $S$  can be represented as a rooted *derivation tree*  $\mathcal{T} = \mathcal{T}(S)$ . Nodes of  $\mathcal{T}$  are identified with blocks. The leaves of  $\mathcal{T}$  correspond to the blocks of  $S$  whereas internal nodes correspond to blocks of derived sequences. Let  $\mathcal{B}(v)$  be the block of  $v \in \mathcal{T}$ , which may be treated as a *set* of symbols if we are indifferent to their permutation in  $\mathcal{B}(v)$ .

*Base Case.* Suppose  $S = B_1 B_2$  is a two block sequence, where each block contains the whole alphabet  $\Sigma(S)$ . The tree  $\mathcal{T}(S)$  consists of three nodes  $u, u_1$ , and  $u_2$ , where  $u$  is the parent of  $u_1$  and  $u_2$ ,  $\mathcal{B}(u_1) = B_1$ ,  $\mathcal{B}(u_2) = B_2$ , and  $\mathcal{B}(u)$  does not exist. For every  $a \in \Sigma(S)$  call  $u$  its *crown* and  $u_1$  and  $u_2$  its left and right *heads*, respectively. These nodes are denoted  $\text{cr}_{|a}$ ,  $\text{lh}_{|a}$ , and  $\text{rh}_{|a}$ .

*Inductive Case.* If  $S$  contains  $m > 2$  blocks, choose a uniform block partition  $\{m_q\}_{1 \leq q \leq \hat{m}}$ , that is, one where  $m_1, \dots, m_{\hat{m}-1}$  are equal powers of two and  $m_{\hat{m}}$  may be smaller. This block partition induces local sequences  $\{\check{S}_q\}_{1 \leq q \leq \hat{m}}$  and an  $\hat{m}$ -block contracted global sequence  $\hat{S}'$ . Inductively construct derivation trees  $\hat{\mathcal{T}} = \mathcal{T}(\hat{S}')$  and  $\{\check{\mathcal{T}}_q\}_{1 \leq q \leq \hat{m}}$ , where  $\check{\mathcal{T}}_q = \mathcal{T}(\check{S}_q)$ . To obtain  $\mathcal{T}(S)$ , identify the root of  $\check{\mathcal{T}}_q$  (which has no block) with the  $q$ th leaf of  $\hat{\mathcal{T}}$ , then place the blocks of  $S$  at the leaves of  $\mathcal{T}$ . This last step is necessary since only local symbols appear in the blocks of  $\{\check{\mathcal{T}}_q\}$  whereas the leaves of  $\mathcal{T}$  must be identified with the blocks of  $S$ . The crown and heads of each symbol  $a \in \Sigma(S)$  are inherited from  $\hat{\mathcal{T}}$ , if  $a$  is global, or some  $\check{\mathcal{T}}_q$  if  $a$  is local to  $S_q$ . See Figure 2 for a schematic.

**5.1. Special Derivation Trees.** It is useful to constrain  $\mathcal{T}$  to use a uniform block partition. Every derivation tree generated in this fashion can be embedded in a full rooted binary tree with height  $\lceil \log m \rceil$ , though the composition of blocks depends

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<sup>9</sup>For example, when  $s = 6$ ,  $\gamma_{r,s-2}(\gamma_{r,s}(n)) = O\left(2^{\alpha\left(2^{\alpha^2(n)/2 + O(\alpha(n))}\right)}\right) = O(2^{\alpha(\alpha(n))})$

is non-constant. Nonetheless  $O(2^{\alpha(\alpha(n))}) \cdot \Lambda_{r,s}(n, 3n) = O(2^{\alpha(\alpha(n))}) \cdot n \cdot 2^{\alpha^2(n)/2 + O(\alpha(n))} = n \cdot 2^{\alpha^2(n)/2 + O(\alpha(n))}$ .

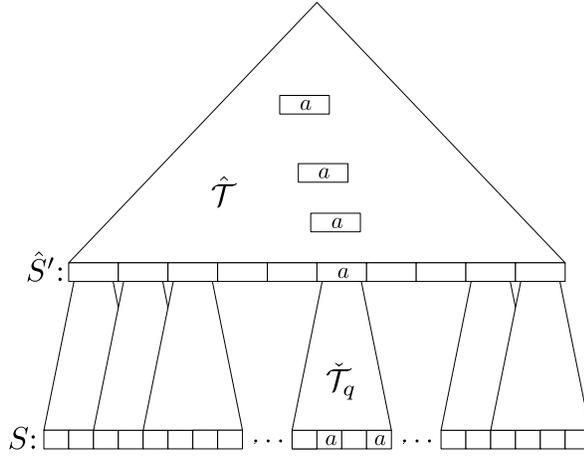


FIG. 2. The derivation tree  $\mathcal{T}(S)$  is the composition of  $\hat{\mathcal{T}} = \mathcal{T}(\hat{S}')$  and  $\{\check{\mathcal{T}}_q\}_{1 \leq q \leq \hat{m}}$ , where  $\check{\mathcal{T}}_q = \mathcal{T}(\check{S}_q)$ . A global symbol  $a \in \Sigma(\hat{S})$  appears in blocks at the leaf level of  $\mathcal{T}$ , at the leaf level of  $\hat{\mathcal{T}}$ , and possibly at higher levels of  $\hat{\mathcal{T}}$ .

on how block partitions are chosen. We will generate two varieties of derivation trees. At one extreme is the *canonical* derivation tree, where block partitions are chosen in the least aggressive way possible. At the other extreme is one where block partitions are guided by Ackermann's function.

*Canonical Derivation Trees.* The canonical derivation tree  $\mathcal{T}^*(S)$  of a sequence  $S$  is obtained by choosing the uniform block partition with  $\hat{m} = \lceil m/2 \rceil$ . We form  $\mathcal{T}^*(S)$  by constructing  $\mathcal{T}^*(\hat{S}')$  recursively and composing it with the trivial three-node base case trees  $\{\mathcal{T}(\check{S}_q)\}_q$ .

*Derivation Trees via Ackermann's Function.* Given a parameter  $i \geq 1$ , define  $j \geq 1$  to be minimal such that  $m \leq a_{i,j}$ . If  $j = 1$  then  $m = a_{i,1} = 2$ , meaning  $\mathcal{T}(S)$  must be the three-node base case tree. When  $j > 1$  we choose a uniform block partition with width  $w = a_{i,j-1}$  (which is a power of 2), so  $\hat{m} = \lceil m/w \rceil \leq a_{i,j}/a_{i,j-1} = a_{i-1,w}$ . The global tree  $\hat{\mathcal{T}}$  is constructed recursively with parameter<sup>10</sup>  $i - 1$  and each local tree  $\check{\mathcal{T}}_q$  is constructed recursively with parameter  $i$ .

**5.2. Projections of the Derivation Tree.** The *projection of  $\mathcal{T}$  onto  $a \in \Sigma(S)$* , written  $\mathcal{T}|_a$ , is the tree rooted at  $\text{cr}|_a$  on the node set  $\{\text{cr}|_a\} \cup \{v \in \mathcal{T} \mid a \in \mathcal{B}(v)\}$ . The edges of  $\mathcal{T}|_a$  represent paths in  $\mathcal{T}$  passing through blocks that do not contain  $a$ .

DEFINITION 5.1. (**Anatomy of a projection tree**)

- The leftmost and rightmost leaves of  $\mathcal{T}|_a$  are wingtips, denoted  $\text{lt}|_a$  and  $\text{rt}|_a$ .
- The left and right wings are those paths in  $\mathcal{T}|_a$  extending from  $\text{lh}|_a$  to  $\text{lt}|_a$  and from  $\text{rh}|_a$  to  $\text{rt}|_a$ .
- Descendants of  $\text{lh}|_a$  and  $\text{rh}|_a$  in  $\mathcal{T}|_a$  are called doves and hawks, respectively.
- A child of a wing node that is not itself on the wing is called a quill.
- A leaf is called a feather if it is the rightmost descendant of a dove quill or leftmost descendant of a hawk quill.

<sup>10</sup>Note that when  $i = 1$  it does not matter that  $i - 1 = 0$  is an invalid parameter. In this case  $w = a_{1,j-1} = a_{1,j}/2$  and  $\hat{m} = 2$ , so  $\hat{\mathcal{T}}$  is forced to be a three-node base case tree.

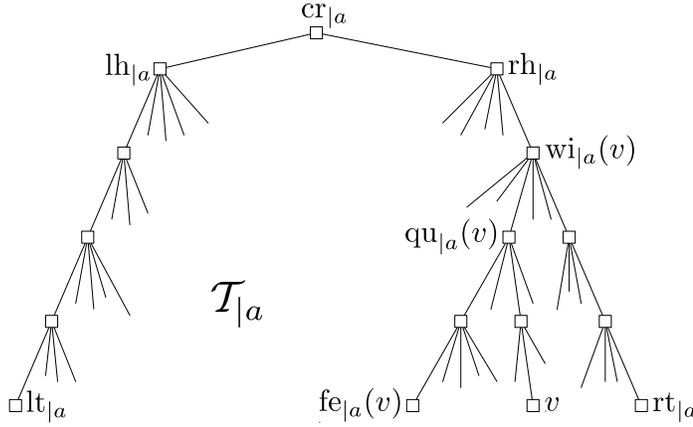


FIG. 3. In this example  $v$  is a hawk leaf in  $\mathcal{T}|_a$  since it is a descendant of  $rh|_a$ . Its wing node  $wi|_a(v)$ , quill  $qu|_a(v)$ , and feather  $fe|_a(v)$  are indicated.

- Suppose  $v$  is a node in  $\mathcal{T}|_a$ . Let  $wi|_a(v)$  be the nearest wing node ancestor of  $v$ ,  $qu|_a(v)$  the quill ancestral to  $v$ , and  $fe|_a(v)$  the feather descending from  $qu|_a(v)$ . See Figure 3 for an illustration.

If  $\mathcal{T}(S)$  is specified, the terms feather and wingtip can also be applied to individual occurrences in  $S$ . For example, an occurrence of  $a$  in block  $\mathcal{B}(v)$  of  $S$  is a feather if  $v$  is a feather in  $\mathcal{T}|_a$ .

When  $\mathcal{T}(S)$  is constructed according to Ackermann’s function, a short proof by induction shows that the height of each projection tree  $\mathcal{T}|_a$  (distance from  $cr|_a$  to a leaf) is at most  $i + 1$ .

**6. Upper Bounds on  $\text{dblForm}(r, 4)$ -free Sequences.** Since order-3 DS sequences are necessarily  $\text{Form}(2, 4)$ -free, we have  $\Lambda_{r,3}^{\text{dbl}}(n) \geq \Lambda_{r,3}(n) \geq \lambda_3(n) = \Theta(n\alpha(n))$ . In this Section we prove tight upper bounds of  $\Lambda_{r,3}^{\text{dbl}}(n) = O(n\alpha(n))$ . These bounds imply  $\lambda_3^{\text{dbl}}(n)$  is also  $O(n\alpha(n))$ , resolving one of Klazar’s open problems [13].

Our analysis is different in character from all previous analyses of (generalized) Davenport–Schinzel sequences. There are two new techniques used in the proof which are worth highlighting. Previous analyses partition the symbols in a block based on some attributes (first, middle, last, etc.), but do not assign any attributes to the blocks themselves. In our analysis we must treat blocks differently based on their context within the larger sequence, that is, according to properties that are independent of the contents of the block. (See the definition of *roosts* in Section 6.2.) The second ingredient is an accounting scheme for bounding the proliferation of symbols. Rather than count the *number* of occurrences of a symbol, say  $b$ , we assign each occurrence of  $b$  a *potential* based on its context. If one  $b$  in  $\hat{S}'$  begets multiple  $bs$  in  $\hat{S}$ , the number of  $bs$  increases, but the *aggregate* potential of the  $bs$  in  $S$  may, in fact, be at most the potential of the originating  $b$  in  $\hat{S}'$ . That is, sometimes proliferating symbols “pay for themselves.” We only need to track changes in sequence potential, not sequence length. Amortizing the analysis in this way lets us account for the proliferation of symbols across *many* levels of the derivation tree, not just between  $\hat{S}'$  and  $S$ .

**6.1. A Potential-Based Recurrence.** Fix a  $\text{dblForm}(r, 4)$ -free sequence  $Z$  and  $i^* \geq 1$ . Define  $j^*$  to be minimal such that its block count  $\llbracket Z \rrbracket \leq a_{i^*, j^*}$  and let

$\mathcal{T} = \mathcal{T}(Z)$  be constructed as in Section 5.1 with parameter  $i^*$ . In this section we analyze a sequence  $S$  encountered in the recursive decomposition of  $Z$ , that is,  $S$  is either  $Z$  itself or a sequence encountered when recursively decomposing  $\hat{Z}'$  and  $\{\check{Z}_q\}$ . Since  $S < Z$ , it too must be  $\text{dblForm}(r, 4)$ -free but we can often say something stronger. If each occurrence of a symbol in  $S$  represents at least two occurrences in  $Z$  then  $S$  must be  $\text{Form}(r, 4)$ -free.<sup>11</sup> Call an occurrence in  $S$  *terminal* if it represents exactly one occurrence in  $Z$  and *non-terminal* otherwise. In terms of the derivation tree, an occurrence of  $a$  in  $S$  is terminal iff it has exactly one leaf descendant in  $\mathcal{T}|_a$ .

Each occurrence of a symbol in  $S$  carries a nonnegative integer potential based on its context within  $S$  and even within  $\mathcal{T}(Z)$ . Since the length of  $S$  is no more than its aggregate potential, it suffices to upper bound the potential. Define  $\Upsilon(n, m)$  to be the maximum potential of an  $m$ -block sequence over an  $n$ -letter alphabet encountered in decomposing  $Z$ . The way potentials are assigned will be discussed shortly. For the time being it suffices to know that the maximum potential is  $\phi = O(1)$ , all terminals carry unit potential, and all non-terminals carry potential at least three.

Our goal is to prove that  $\Upsilon$  obeys the following recurrence.

RECURRENCE 6.1.

$$\begin{aligned} \Upsilon(n, m) = & \sum_{1 \leq q \leq \hat{m}} \Upsilon(\check{n}_q, m_q) + 2 \cdot \left[ \phi \cdot \Lambda_{r,2}(\hat{n}, m) + \Lambda_{r,2}^{\text{dbl}}(\hat{n}, m) + \hat{n} \right] + \Upsilon(\hat{n}, \hat{m}) \\ & + (r-1)\phi \cdot m + 2[(r-1)(i^* - 2)]^2 \cdot \hat{m} \end{aligned}$$

Decomposing  $S$  as usual, it follows that the maximum potential of local sequences  $\{\check{S}_q\}_q$  is  $\sum_q \Upsilon(\check{n}_q, m_q)$ , giving the first term of Recurrence 6.1. The sequence  $\hat{S}$  of global first occurrences can be partitioned into terminals  $\hat{S}^t$  and non-terminals  $\hat{S}^{\text{nt}}$ . After removing the last occurrence of each symbol in  $\hat{S}^t$ , the resulting sequence is  $\text{dblForm}(r, 3)$ -free, so its length (and potential) is  $|\hat{S}^t| \leq \Lambda_{r,2}^{\text{dbl}}(\hat{n}, m) + \hat{n}$ . We endow each non-terminal in  $\hat{S}^{\text{nt}}$  an initial potential at most  $\phi$ . (Note that occurrences of  $a$  in  $\hat{S}$  correspond to quills in  $\mathcal{T}|_a$ .) Being  $\text{Form}(r, 3)$ -free, the potential of  $\hat{S}^{\text{nt}}$  is therefore at most  $\phi \cdot \Lambda_{r,2}(\hat{n}, m)$ . A symmetric analysis is applied to  $\check{S}$ , the sequences of last occurrences, which gives the second term of Recurrence 6.1.

The global contracted sequence  $\hat{S}'$  begets  $\hat{S}$ ,  $\check{S}$ , and  $\bar{S}$ , the first two of which we have just accounted for. In general  $|\bar{S}|$  may be significantly larger than  $|\hat{S}'|$ . We account for this proliferation in symbols by showing that the aggregate potential of  $\bar{S}$  is nonetheless at most that of  $\hat{S}'$  plus  $(r-1)\phi \cdot m + 2[(r-1)(i^* - 1)]^2 \cdot \hat{m}$ , which explains the last three terms of Recurrence 6.1. Consider the sequence  $\bar{S}_q$  begat by the middle symbols of block  $B_q$  in  $\hat{S}'$ . We decompose  $\bar{S}_q$  as follows.

1. Tag any symbol occurring exactly once in  $\bar{S}_q$ . (Its potential in  $\bar{S}_q$  will be at most its potential in  $\hat{S}'$ .)
2. Tag the first non-terminal occurrence of each symbol in  $\bar{S}_q$ .
3. Tag the first, second, and last terminal occurrence of each symbol in  $\bar{S}_q$ .
4. Tag the first  $r-1$  untagged occurrences (terminal and non-terminal) in each block of  $\bar{S}_q$ .

Symbols that are tagged in both of Steps 2 and 3 have *molted*; all others are *unmolted*. We will say that the non-terminal  $a$  tagged in Step 2 has *molted* those terminal  $a$  tagged in Step 3. See Figure 4 for a schematic.

<sup>11</sup>This is not quite true, but we can make this inference when bounding  $\Lambda_{r,3}^{\text{dbl}}$  asymptotically. See Remark 4.2 for a discussion of this issue.

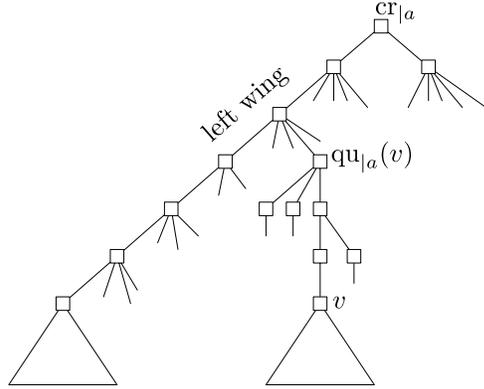


FIG. 4. Here  $v$  is an internal node of  $\mathcal{T}|_a$ . Between  $\text{qu}|_a(v)$  and  $v$ ,  $a$  has molted twice: at  $v$ 's parent it molted one  $a$  to the right and at  $v$ 's grandparent it molted two  $a$ s to the left.

We claim  $\bar{S}_q$  has been completely tagged after Step 4. If this were not so, there must be  $r$  symbols  $a_1, \dots, a_r$  in some block  $B$  in  $\bar{S}_q$ . If  $a_k$  is terminal in  $B$  it must be preceded by two terminal  $a_k$ s and followed by one terminal  $a_k$  in  $\bar{S}_q$ ; if  $a_k$  is non-terminal in  $B$  it must be preceded by a non-terminal  $a_k$ . Dividing  $\bar{S}_q$  at the left boundary of  $B$ , we see two occurrences of each of  $a_1, \dots, a_r$  on both the left and right side of the boundary, which may take the form of one non-terminal or two terminals. Since  $a_1, \dots, a_k$  are categorized as global middle in  $S_q$ , each appears both before and after  $S_q$ , yielding an instance of  $\text{dblForm}(r, 4)$  in  $Z$ , a contradiction.

The aggregate potential of those symbols tagged in Step 4 is at most  $(r-1)\phi \cdot m$ , which are covered by the second-to-last term of Recurrence 6.1. Suppose that  $a \in B_q$  is non-terminal in  $\bar{S}'$  but it begets only terminal  $a$ s in  $\bar{S}_q$ , that is, no  $a$ s are tagged in Step 2. This proliferation of  $a$ s causes no net increase in potential since the  $a \in B_q$  carries potential at least 3, which covers the potential of the three terminal  $a$ s tagged in Step 3. In general, for each molted symbol  $a$ , we will tag one non-terminal and up to three terminals in Steps 2 and 3. This will cause no net increase in potential *provided* that the  $a$  in  $B_q$  carries at least the potential of the non-terminal  $a$  in  $\bar{S}_q$  plus 3. In order to avoid cumbersome statements, we will treat the non-terminal  $a$  tagged in Step 2 as the “same”  $a \in B_q$ . For example, if  $B$  is a block in  $\bar{S}_q$  and  $a \in B$  is non-terminal, to say *the  $a \in B$  has molted four times* means that, in  $\mathcal{T}|_a$ ,  $B$  has four ancestors, possibly including itself, and all strict descendants of  $\text{qu}|_a(B)$ , which each have at least one sibling in  $\mathcal{T}|_a$ . This sibling corresponds to an  $a$  removed in Step 3 at some stage in the decomposition of  $S$ .

In the remainder of this section we explain why it suffices to endow each new non-terminal quill with a *constant* potential  $\phi$ . The analysis above shows that  $3 \cdot (i^* - 1)$  suffices, which is not constant.<sup>12</sup>

<sup>12</sup>Observe that for any  $a \in \Sigma(Z)$ , the height of  $\mathcal{T}|_a$  is  $i^* + 1$  and all quills of  $\mathcal{T}|_a$  are at distance at least 2 from  $\text{cr}|_a$ . Every non-terminal quill can therefore molt up to  $i^* - 1$  times, generating up to three terminals per molting, each of which carries unit potential.

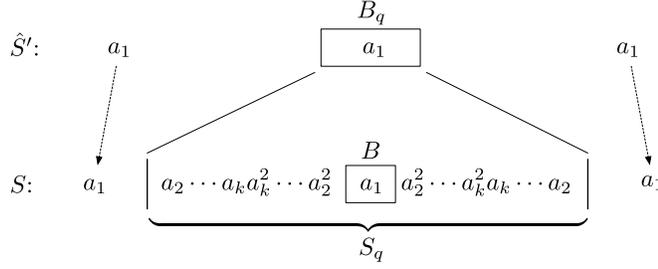


FIG. 5. A  $k$ -egg is formed when a middle  $a_1 \in B_q$  is dropped into a  $(k-1)$ -root in  $\tilde{S}_q$ .

**6.2. Roosts, Eggs, and Fertility.** Our analysis considers properties of blocks (and of occurrences of symbols) that depend on their context within a larger sequence.

**DEFINITION 6.2. (Roosts and Eggs)** Let  $S$  be a sequence encountered in the decomposition of  $Z$ .

1. An interval  $I$  of zero or more blocks in  $S$  is a  $k$ -roost if there are  $k$  distinct symbols  $a_1, \dots, a_k$  such that the sequence contains

$$a_1 a_2 \cdots a_k \ a_k^2 a_{k-1}^2 \cdots a_1^2 \ I \ a_1^2 a_2^2 \cdots a_k^2 \ a_k a_{k-1} \cdots a_1,$$

where  $b^2$  refers to two terminal  $b$ s or one non-terminal  $b$ . The occurrences of  $a_1$  just to the left and right of  $I$  are called  $k$ -left mature and  $k$ -right mature. A  $k$ -mature occurrence of a symbol whose block is a  $k$ -roost is infertile. A  $k$ -left mature occurrence that is not infertile is  $k$ -left fertile;  $k$ -right fertile is defined analogously. (For any  $l < k$ ,  $k$ -roosts are clearly also  $l$ -roosts, and  $k$ -mature occurrences also  $l$ -mature.)

2. An occurrence of  $a_1$  in block  $B$  of  $S$  is a  $k$ -egg if the sequence contains

$$a_1 a_2 \cdots a_k \ a_k^2 a_{k-1}^2 \cdots a_2^2 \ B \ a_2^2 a_3^2 \cdots a_k^2 \ a_k a_{k-1} \cdots a_1$$

Note that any middle occurrence of a symbol is a 1-egg.

One may already discern from Definition 6.2 the shape of the rest of the proof. A  $k$ -roost can only exist if the sequence contains a  $\text{dblForm}(k, 4)$  sequence, so there cannot be  $r$ -roosts. If the proliferation of symbols necessarily leads to  $k$ -roosts for ever larger  $k$ , we have a cap on the proliferation of symbols. Lemma 6.3 lists some straightforward consequences of Definition 6.2.

**LEMMA 6.3. (Properties of Roosts and Eggs)** Let  $S$  be an  $m$ -block sequence encountered in the recursive decomposition of a  $\text{dblForm}(r, 4)$ -free sequence  $Z$ . Define  $\{S_q, \tilde{S}_q, \hat{S}_q\}_{1 \leq q \leq \hat{m}}$  and  $\hat{S}' = B_1 \cdots B_{\hat{m}}$  as usual.

1. No block in  $S$  is an  $r$ -roost. All  $r$ -eggs represent at most 3 occurrences in  $Z$ .
2. If  $B_q$  is a  $k$ -roost in  $\hat{S}'$ , every block of  $S_q$  is a  $k$ -roost in  $S$ .
3. Let  $B$  be a block in  $S_q$  containing a global symbol  $a$ . If  $B$  is a  $(k-1)$ -roost in  $\tilde{S}_q$  and the  $a \in B_q$  is a middle occurrence in  $\hat{S}'$  then  $a \in B$  is a  $k$ -egg in  $S$ . See Figure 5.
4. Let  $B$  be a block in  $S_q$  containing a global symbol  $a$ . Suppose the  $a \in B_q$  is  $k$ -left fertile in  $\hat{S}'$  and the  $a \in B$  is  $k$ -left fertile in  $S$ . All blocks following  $B$  in  $S_q$  are  $k$ -roosts in  $S$ . A symmetric statement is true of  $k$ -right fertile occurrences. See Figure 6.

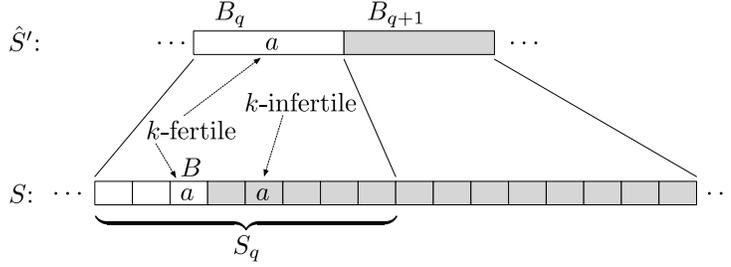


FIG. 6. The shaded blocks are  $k$ -roosts. A  $k$ -left fertile occurrence of  $a \in B_q$  in  $\hat{S}'$  begets at most one  $k$ -left fertile occurrence in  $S_q$ , and, in this example, one  $k$ -infertile occurrence. Since  $B_{q+1}$  is a  $k$ -roost in  $\hat{S}'$ , all blocks in  $S_{q+1}$  are  $k$ -roosts in  $S$  whether or not they were already  $k$ -roosts in  $\hat{S}_{q+1}$ .

**6.3. Molting and the Evolution of Potentials.** Consider the status of a non-terminal symbol  $a$  as it descends, in  $\mathcal{T}|_a$ , from  $\text{qu}|_a(v)$  to some leaf  $v$ . Since  $a \in \mathcal{B}(\text{qu}|_a(v))$  is a middle symbol at that level (it is not on either wing of  $\mathcal{T}|_a$ ), this  $a$  begins as a 1-egg and may become 1-fertile (left or right), then 1-infertile, then a 2-egg, 2-fertile, 2-infertile, and so on. It cannot become  $r$ -mature (fertile or infertile) for this would mean that  $\text{dblForm}(r, 4) < Z$ , so there are at most  $3(r-1)$  transitions. Multiple transitions may occur simultaneously. When a non-terminal first becomes a  $k$ -egg, or  $k$ -fertile, or  $k$ -infertile, its potential becomes  $\phi_k^{\text{eg}}$ ,  $\phi_k^{\text{fe}}$ , or  $\phi_k^{\text{in}}$ , where

$$\phi = \phi_1^{\text{eg}} > \phi_1^{\text{fe}} > \phi_1^{\text{in}} > \cdots > \phi_{r-1}^{\text{eg}} > \phi_{r-1}^{\text{fe}} > \phi_{r-1}^{\text{in}} > \phi_r^{\text{eg}} = 3$$

If we can show that each symbol molts  $O(1)$  times between status transitions, it suffices to set the initial potential at  $\phi = O(r) = O(1)$ . This is clearly true of  $k$ -egg  $\rightarrow k$ -mature transitions. Any  $k$ -egg  $a$  that molts three  $a$ s must have molted two of them to the same side, left or right, making it  $k$ -mature. Since a non-terminal can molt up to 3 terminals in the molting event that makes it  $k$ -mature, it suffices to set  $\phi_k^{\text{eg}} - \phi_k^{\text{fe}} = 5$ . (If this  $a$  transitions directly from a  $k$ -egg to  $k$ -infertile, all the better, for  $\phi_k^{\text{in}} < \phi_k^{\text{fe}}$ .) We now analyze the  $k$ -fertile  $\rightarrow k$ -infertile and  $k$ -infertile  $\rightarrow (k+1)$ -egg transitions.

**LEMMA 6.4.** Fix a block index  $q \leq \llbracket \hat{S}' \rrbracket$  and let  $F \subset B_q$  be those symbols newly  $k$ -left fertile, that is, they were not  $k$ -left fertile at any ancestor of  $B_q$  in their respective derivation trees. The total number of terminals molted by  $F$ -symbols before they become  $k$ -infertile is at most  $2|F| + (r-1)\binom{i^*-1}{2}$ .

*Proof.* Lemma 6.3(4) implies that so long as symbols in  $F$  remain  $k$ -fertile, as they travel from  $B_q$  to a block in  $S_q$ , to blocks at lower levels of the derivation tree, they will always be contained in a *single* block at that level of the tree. In other words, there is a sequence of nodes  $(B_q = v_1, v_2, \dots, v_l)$  in  $\mathcal{T}$  lying on a path from  $B_q = v_1$  (in  $\hat{S}'$ ), to  $v_2$  (in  $S$ ), to a descendant leaf  $v_l$  (where  $l \leq i^*$ ) such that any symbol  $a \in F$  is  $k$ -left fertile in some *prefix* of the list  $\mathcal{B}(v_1), \mathcal{B}(v_2), \dots, \mathcal{B}(v_l)$ . See Figure 7. Call a symbol  $a \in F$  *type*  $(f, g)$  if  $a$  molted a terminal to the right at both  $\mathcal{B}(v_f)$  and  $\mathcal{B}(v_g)$ , for  $1 < f < g \leq l$ .<sup>13</sup> That is, in  $\mathcal{T}|_a$ ,  $\mathcal{B}(v_f)$  and  $\mathcal{B}(v_g)$  have right siblings. Note that during the time in which this  $a$  is  $k$ -left fertile it can molt at most once to the left: molting two  $a$ s to the left would make it  $k$ -infertile.

<sup>13</sup>Note that a symbol that molts exactly twice to the right has one type. In general, a symbol that molts  $h$  times to the right is of  $\binom{h}{2}$  distinct types.

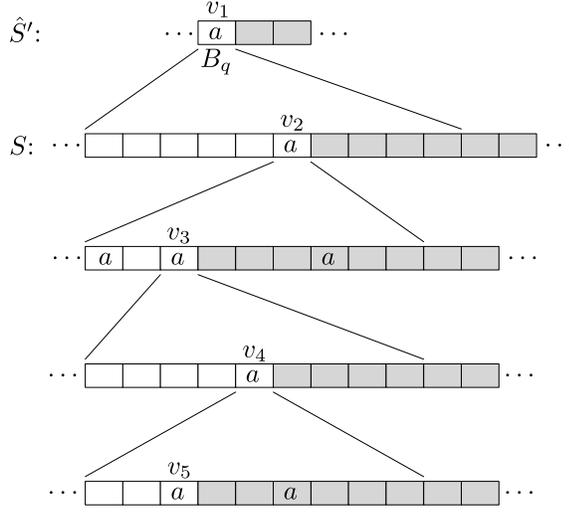


FIG. 7. A newly  $k$ -left-fertile symbol  $a \in B_q = \mathcal{B}(v_1)$  in  $\hat{S}'$ . As  $a$  progresses down  $\mathcal{T}_a$  it continues to be  $k$ -left fertile at  $\mathcal{B}(v_2), \dots, \mathcal{B}(v_5)$ . Since it molts to the right at blocks  $\mathcal{B}(v_3)$  and  $\mathcal{B}(v_5)$  it has type  $(3, 5)$ . It also molts to the left at  $\mathcal{B}(v_3)$ . Were it to molt twice to the left at  $\mathcal{B}(v_3)$ ,  $\mathcal{B}(v_3)$  would then become a  $k$ -roost and the  $a \in \mathcal{B}(v_3)$   $k$ -infertile.

By the pigeonhole principle, if  $(r-1)\binom{i^*-1}{2} + 1$  symbols in  $F$  molted twice to the right then a subset  $F' \subset F$  of  $r$  of them have the same type, say  $(f, g)$ . However, this would imply that  $Z$  is not  $\text{dblForm}(r, 4)$ -free. Since  $k$ -fertile symbols are middle symbols, every symbol in  $F'$  appears at least once before and after  $B_q$ . The occurrences of  $F'$ -symbols in  $\mathcal{B}(v_g)$  are non-terminal, so they each represent at least two occurrences in  $Z$ . Finally, the  $F'$ -symbols appear twice at descendants of  $B_q$  but to the right of  $\mathcal{B}(v_g)$ . See Figure 7.

To sum up, we let each  $F$ -symbol molt once to the left and once to the right while  $k$ -left fertile. Some subset can molt more than once to the right, but the total number of such terminals molted by these symbols is at most  $(r-1)\binom{i^*-1}{2}$ .  $\square$

A nearly symmetric analysis can be applied to right fertile symbols. The asymmetry comes from the fact that non-terminals can molt two terminals to the left but only one to the right.

LEMMA 6.5. Fix a block index  $q \leq \llbracket \hat{S}' \rrbracket$  and let  $F \subset B_q$  be those symbols newly  $k$ -right fertile, that is, they were not  $k$ -left fertile at any ancestor of  $B_q$  in their respective derivation trees. The total number of terminals molted by  $F$ -symbols before they become  $k$ -infertile is at most  $2|F| + (r-1)\left(\binom{i^*-1}{2} + i^* - 1\right)$ .

*Proof.* The argument is the same as above, except that we allow types  $(f, f)$  if a symbol molts twice to the left at  $\mathcal{B}(v_f)$ . There are now at most  $\left(\binom{i^*-1}{2} + i^* - 1\right)$  possible types, and we cannot see  $r$  symbols of the same type.  $\square$

According to Lemmas 6.4 and 6.5, it suffices to set  $\phi_k^{\text{fe}} = \phi_k^{\text{in}} + 2$ . The total number of molted terminals unaccounted for, over all  $q$ , all  $k < r$ , counting both  $k$ -left fertile and  $k$ -right fertile symbols in  $B_q$ , is  $\hat{m} \cdot (r-1)^2 \left(2\binom{i^*-1}{2} + i^* - 1\right) < \hat{m} \cdot [(r-1)(i^* - 1)]^2$ , which are covered by the last term of Recurrence 6.1.

The remaining task is to analyze the  $k$ -infertile  $\rightarrow (k+1)$ -egg transition.

LEMMA 6.6. Let  $u, v, w$  be distinct nodes such that  $a, b \in \mathcal{B}(u)$ ,  $a \in \mathcal{B}(v)$ ,  $b \in \mathcal{B}(w)$ , where  $v$  is the parent of  $u$  in  $\mathcal{T}_a$  and  $w$  is the parent of  $u$  in  $\mathcal{T}_b$ . If  $a, b$  were  $k$ -infertile

in blocks  $\mathcal{B}(v)$  and  $\mathcal{B}(w)$  then at least one of  $a, b$  became a  $(k+1)$ -egg when it was inserted into  $\mathcal{B}(u)$ .

*Proof.* This is a consequence of Lemma 6.3(2,3). Without loss of generality  $w$  is a strict ancestor of  $v$ , so  $a$  was inserted into  $\mathcal{B}(u)$  before  $b$  was inserted into  $\mathcal{B}(u)$ . Since the  $a \in \mathcal{B}(v)$  was  $k$ -infertile,  $\mathcal{B}(v)$  was a  $k$ -roost, by definition. By Lemma 6.3(2),  $\mathcal{B}(u)$  became a  $k$ -roost after  $a$  was inserted there. By Lemma 6.3(3), when  $b$  was inserted in  $\mathcal{B}(u)$  it became a  $(k+1)$ -egg.  $\square$

LEMMA 6.7. *Let  $I \subset \Sigma(\hat{S}_q)$  be those non-terminals that were  $k$ -infertile, non- $(k+1)$ -eggs in  $B_q$  but became  $(k+1)$ -eggs in  $S_q$ . The number of terminals molted by  $I$  symbols while they were  $k$ -infertile, non- $(k+1)$ -eggs is at most  $2|I| + (r-1)(2^{\binom{i^*-2}{2}} + i^* - 2)$ .*

*Proof.* Lemma 6.6 implies that on a path from  $B_q$  to the root of  $\mathcal{T}$  we encounter nodes  $v_1 = B_q, v_2, \dots, v_l$ , not necessarily adjacent, such that, for each symbol  $a \in I$ , the set of blocks in which  $a$  is  $k$ -infertile and not a quill is some prefix of  $\mathcal{B}(v_1), \dots, \mathcal{B}(v_l)$ , where  $l \leq i^* - 2$ . Call an  $a \in I$  type  $(\rightarrow, f, g)$  if it molted a terminal to the right in both  $\mathcal{B}(v_f)$  and  $\mathcal{B}(v_g)$ , where  $1 \leq f < g \leq l$ . Call it type  $(\leftarrow, f, g)$ , where  $1 \leq f \leq g \leq l$ , if it molted a terminal to the left in both  $\mathcal{B}(v_f)$  and  $\mathcal{B}(v_g)$ , or two terminals to the left if  $f = g$ . There are  $2^{\binom{l}{2}} + l$  distinct types. There cannot be  $r$  symbols of one type, for this would imply that  $Z$  is not  $\text{dblForm}(r, 4)$ -free. (The argument is the same as in the proof of Lemma 6.4.) Since every symbol that molts more than two terminals is of at least one type, the total number of terminals molted by  $I$  while being  $k$ -infertile, non- $(k+1)$ -eggs is  $2|I| + (r-1)(2^{\binom{i^*-2}{2}} + i^* - 2)$ .  $\square$

We set  $\phi_k^{\text{in}} - \phi_{k+1}^{\text{eg}} = 2$ , so the total number of terminals unaccounted for, over all  $q < \hat{m}$  and  $k < r$ , is at most  $\hat{m} \cdot [(r-1)(i^* - 2)]^2$ , which is covered by the last term of Recurrence 6.1. Given the constraints we have established on potentials it suffices to set  $\phi = \phi_1^{\text{eg}} = 7(r-1) + 1$ , since  $|\phi_k^{\text{eg}} - \phi_k^{\text{fe}}| = 5$ ,  $|\phi_k^{\text{fe}} - \phi_k^{\text{in}}| = |\phi_k^{\text{in}} - \phi_{k+1}^{\text{eg}}| = 2$ , and  $\phi_r^{\text{eg}} = 3$ .

REMARK 6.8. *Observe the asymmetry in the arguments of Lemmas 6.4–6.5 and Lemma 6.7. In Lemmas 6.4 and 6.5 we are tracking moltings that will happen “in the future” (below the level of  $S$  in  $\mathcal{T}$ ) whereas in Lemma 6.7 we are accounting for moltings that have already occurred at and above the level of  $\hat{S}'$  in  $\mathcal{T}$ .*

**6.4. Wrapping Up the Analysis.** Since  $\Lambda_{r,2}(\cdot, \cdot)$  and  $\Lambda_{r,2}^{\text{dbl}}(\cdot, \cdot)$  are both linear and  $\hat{m} < m$ , we can simplify Recurrence 6.1 to

$$\Upsilon(n, m) \leq \sum_{1 \leq q \leq \hat{m}} \Upsilon(\check{n}_q, m_q) + \Upsilon(\hat{n}, \hat{m}) + C[\hat{n} + (i^*)^2 m],$$

for some constant  $C$  depending only on  $r$ . A straightforward proof by induction shows that for any  $i \leq i^*$  and  $j$  minimal such that  $m \leq a_{i,j}$ ,  $\Upsilon(n, m) \leq Ci(n + (i^*)^2 jm)$ . Putting it all together we have, for  $\|Z\| = n^*$  and  $\llbracket Z \rrbracket = m^*$ ,

$$|Z| \leq \Lambda_{r,3}^{\text{dbl}}(n^*, m^*) \leq \Upsilon(n^*, m^*) \leq Ci^* n^* + C(i^*)^3 j^* m^*. \quad (6.1)$$

Eqn. (6.1) leads to an upper bound of  $\Lambda_{r,3}^{\text{dbl}}(n, m) = O(n\alpha(n, m) + m\alpha^3(n, m))$ , which, by Lemma 3.1, implies an upper bound of  $\Lambda_{r,3}^{\text{dbl}}(n) = O(n\alpha^3(n))$ . Theorem 6.9 reduces this to  $O(n\alpha(n))$ , which is asymptotically tight since  $\Lambda_{r,3}^{\text{dbl}}(n) = \Omega(\lambda_3(n))$ .

THEOREM 6.9. *For any  $r \geq 2$ ,  $\Lambda_{r,3}^{\text{dbl}}(n) = \Theta(n\alpha(n))$  and  $\Lambda_{r,3}^{\text{dbl}}(n, m) = \Theta(n\alpha(n, m) + m)$ .*

*Proof.* Let  $S$  be a  $\text{dblForm}(r, 4)$ -free sequence. To bound  $|S|$  asymptotically we can assume, using Lemmas 3.1 and 3.3, that  $S$  consists of  $m \leq 2n$  blocks. (If there

are  $m > 2n$  blocks, remove up to  $r - 1$  symbols at block boundaries to make it  $r$ -sparse. If the sequence is  $r$ -sparse, we can discard a constant fraction of occurrences to partition the sequence into  $2n$  blocks.) Choose  $i$  to be minimal such that  $m \leq a_{i,j}$ , where  $j = \max\{3, \lceil n/m \rceil\}$ . Partition  $S = S_1 \cdots S_{\hat{m}}$  into  $\hat{m} = \lceil m/i^2 \rceil$  intervals, each consisting of  $i^2$  blocks. Define  $\hat{S}, \hat{S}', \hat{S}_q$ , etc. as usual. Applying Eqn. (6.1) with  $i^* = i$ , we have  $|\hat{S}'| \leq C(i\hat{n} + i^3 j \hat{m}) \leq C(i(\hat{n} + jm)) = O(in)$ . Since each  $\hat{S}_q, \check{S}_q$ , and  $\bar{S}_q$  is  $\text{dblForm}(r, 3)$ -free and  $\Lambda_{r,2}^{\text{dbl}}(n_q, m_q) = O(n_q + m_q)$  is linear, it follows that  $|\hat{S}| = O(in + m) = O(in)$ . We now apply Eqn. (6.1) to local symbols with  $i^* = 1$ , that is, for each index  $q \leq \hat{m}$ ,  $j$  is chosen to be minimal such that  $m_q \leq a_{1,j}$ . Since  $a_{1,j} = 2^j$ ,  $j = \lceil \log m_q \rceil \leq \lceil \log i^2 \rceil$ . It follows that  $|\check{S}| = \sum_q |\check{S}_q| \leq \sum_q C(\tilde{n}_q + m_q \log m_q) = O(\tilde{n} + m \log(i^2)) = O(n \log i)$ . Since  $i = \alpha(n, m) + O(1)$ ,  $|S| = |\hat{S}| + |\check{S}| = O(n\alpha(n, m)) = O(n\alpha(n))$ .  $\square$

Theorem 6.9 and Lemma 1.2 immediately give us asymptotically sharp bounds on the extremal functions for certain doubled forbidden sequences.

**COROLLARY 6.10.** (See Nivasch [16, Rem. 5.1], Pettie [21], Geneson, Prasad, and Tidor [8], and Klazar [13, p. 13].)

$$\begin{aligned} \lambda_3^{\text{dbl}}(n) &= \Theta(\Lambda_{2,3}^{\text{dbl}}(n)) = \Theta(n\alpha(n)), \\ \text{Ex}(\text{dbl}(abcacbc), n) &= \Theta(\Lambda_{4,3}^{\text{dbl}}(n)) = \Theta(n\alpha(n)), && \text{See [21]} \\ \text{Ex}(\text{dbl}(abcabca), n) &= \Theta(\Lambda_{3,3}^{\text{dbl}}(n)) = \Theta(n\alpha(n)), && \text{See [16]} \end{aligned}$$

and, more generally,

$$\text{Ex}(\text{dbl}(1 \cdots k 1 \cdots k 1), n) = \Theta(\Lambda_{r,3}^{\text{dbl}}(n)) = \Theta(n\alpha(n)),$$

where  $r = (k - 1)^3 + 1$ .

**7. Double Davenport–Schinzel Sequences.** Recall from Section 5.1 that the canonical derivation tree  $\mathcal{T}^*(S)$  is obtained by decomposing  $S$  in the least aggressive way possible, choosing  $\hat{m} = \lceil \lceil S \rceil / 2 \rceil$  whenever  $\lceil S \rceil > 2$ . Figure 8 gives an example of such a tree.

The structure of the canonical derivation tree is, in many respects, simpler than general derivation trees. For example, all wing nodes in any projection tree  $\mathcal{T}|_a$ , where  $a \in \Sigma(S)$ , have either one or two children. Those with two children (*branching* nodes) are associated with precisely one quill and therefore one feather,<sup>14</sup> so counting the number of feathers is tantamount to counting branching wing nodes.

Nesting was a concept introduced in [17] to analyze odd-order DS sequences. Here we generalize it to deal with double DS sequences.

**DEFINITION 7.1. (Nesting)** Let  $B$  be a block of  $S$  containing  $a, b \in \Sigma(S)$ . If  $S$  contains either

$$abbBbba \quad \text{or} \quad baabBaba$$

<sup>14</sup>Recall that a *feather* of  $\mathcal{T}|_a$  is the rightmost descendant of a dove quill or leftmost descendant of a hawk quill.

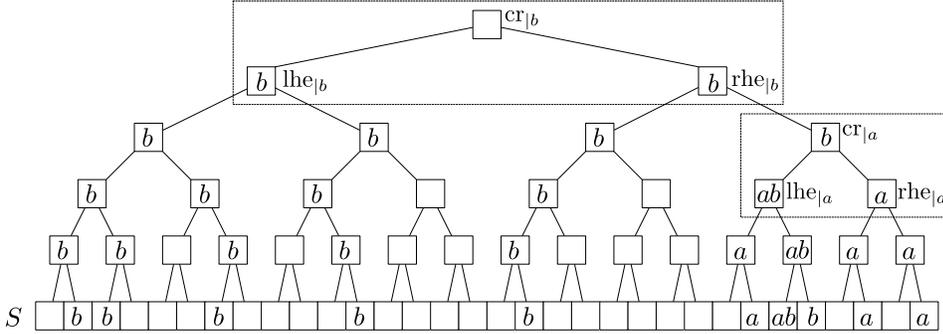


FIG. 8. An example of a canonical derivation tree for  $S$ . Dashed boxes isolate the base case trees that assign  $a, b \in \Sigma(S)$  their crowns and heads.

then  $a$  and  $b$  are called double-nested in  $B$ .

Lemma 7.2 can be thought of as a generalization of [17, Lem. 4.4] to deal with double-nestedness. Whereas [17, Lem. 4.4] assumed any derivation tree, Lemma 7.2 refers to the canonical derivation tree  $\mathcal{T}^*(S')$  as this makes the proof slightly simpler. This assumption is actually without much loss of generality since any derivation tree obtained with uniform block partitions is “contained” in the canonical derivation tree, that is, its blocks are subsequences of the corresponding blocks in the canonical tree.

LEMMA 7.2. Consider a sequence  $S'$ , its canonical derivation tree  $\mathcal{T}^*(S')$ , and a leaf  $v$  for which  $a, b \in \mathcal{B}(v)$ . Let  $S$  be obtained from  $S'$  by substituting, for each leaf  $u \neq v$ , a sequence  $S(u)$  containing at least two copies of each symbol in  $\mathcal{B}(u)$ . (The block  $\mathcal{B}(v)$  appears verbatim in  $S$ .) If  $v$  is neither a wingtip nor feather in both  $\mathcal{T}_{|a}^*$  and  $\mathcal{T}_{|b}^*$  then, in  $S$ ,  $a$  and  $b$  are double-nested in  $\mathcal{B}(v)$ .

*Proof.* Without loss of generality we can assume that  $v$  is a dove in  $\mathcal{T}_{|a}^*$  and  $cr|_b$  is ancestral to  $cr|_a$ . Because  $v$  is neither a wingtip nor feather in  $\mathcal{T}_{|a}^*$ , it must be distinct from the leftmost and rightmost leaf descendants of  $wi|_a(v)$ , namely  $lt|_a$  and  $fe|_a(v)$ . Moreover, since  $v$  is a dove in  $\mathcal{T}_{|a}^*$  it descends from the right child of  $wi|_a(v)$ , namely  $qu|_a(v)$ . Partition  $S$  into four intervals

$I_1$  : everything preceding  $\mathcal{B}(lt|_a)$ .

$I_2$  : everything from  $I_1$  to the beginning of  $\mathcal{B}(v)$ .

$I_3$  : everything from the end of  $\mathcal{B}(v)$  to the end of  $\mathcal{B}(fe|_a(v))$ .

$I_4$  : everything following  $I_3$ .

If  $b$  appeared in both  $I_1$  and  $I_4$  then  $a, b \in \mathcal{B}(v)$  would clearly be double-nested in  $S$ . Therefore it suffices to consider two cases, (1)  $I_1$  contains no  $bs$ , and (2)  $I_4$  contains no  $bs$ . Figures 9 and 10 illustrate the two cases.

*Case 1.* The wingtip  $lt|_b$  must be in interval  $I_2$ , though it may be identical to  $lt|_a$ . Since  $wi|_a(v)$  is ancestral to both  $lt|_b$  and  $v$ , and is a strict descendant of  $cr|_b$ , it follows that  $v$  is a dove in  $\mathcal{T}_{|b}^*$  and that  $wi|_b(v)$  is a descendant of  $wi|_a(v)$ . The rightmost descendant of  $wi|_b(v)$  in  $\mathcal{T}_{|b}^*$  is  $fe|_b(v)$ , which is distinct from  $v$ . Since  $wi|_a(v)$  is a descendant of  $lh|_a$ , any descendant of  $rh|_a$ , such as  $rt|_a$ , lies to the right of  $fe|_b(v)$ , in interval  $I_4$ . By the same reasoning,  $rt|_b$  lies in  $I_4$ .

Regardless of whether  $lt|_a$  and  $lt|_b$  are identical or distinct,  $\mathcal{B}(v)$  is preceded, in  $S$ , by either  $abb$  or  $baa$ . In the first case  $lt|_a, lt|_b, v, fe|_b(v), rt|_a$  certify that  $a, b$  are double-nested in  $\mathcal{B}(v)$ ; see Figure 9. In the latter case  $lt|_b = lt|_a, v, fe|_a(v), rt|_b$  certify that  $a, b$  are double-nested in  $\mathcal{B}(v)$ .

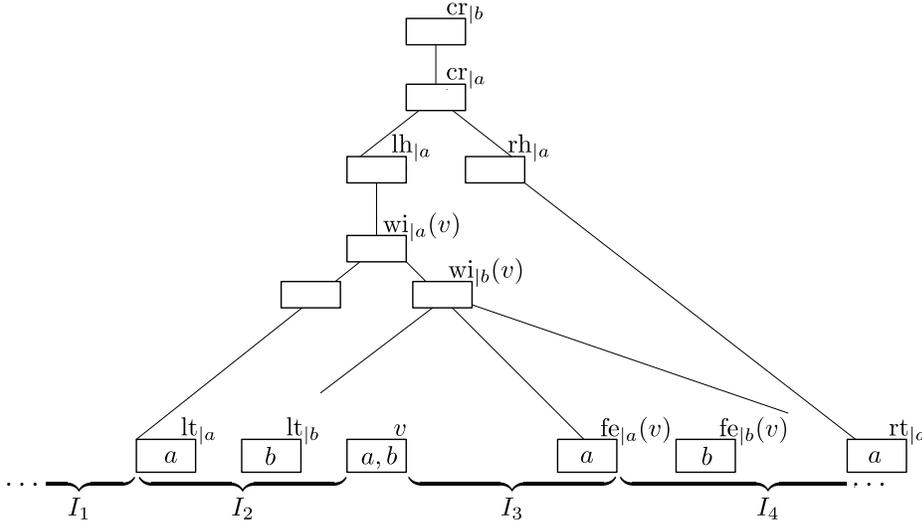


FIG. 9. In Case 1 interval  $I_1$  contains no  $bs$ . Contrary to the depiction,  $lt|_a$  and  $lt|_b$  are not necessarily distinct, nor are  $wi|_a(v)$  and  $wi|_b(v)$  or  $cr|_a$  and  $cr|_b$ . In this depiction  $qu|_a(v)$ , the right child of  $wi|_a(v)$ , happens to be identical to  $wi|_b(v)$ .

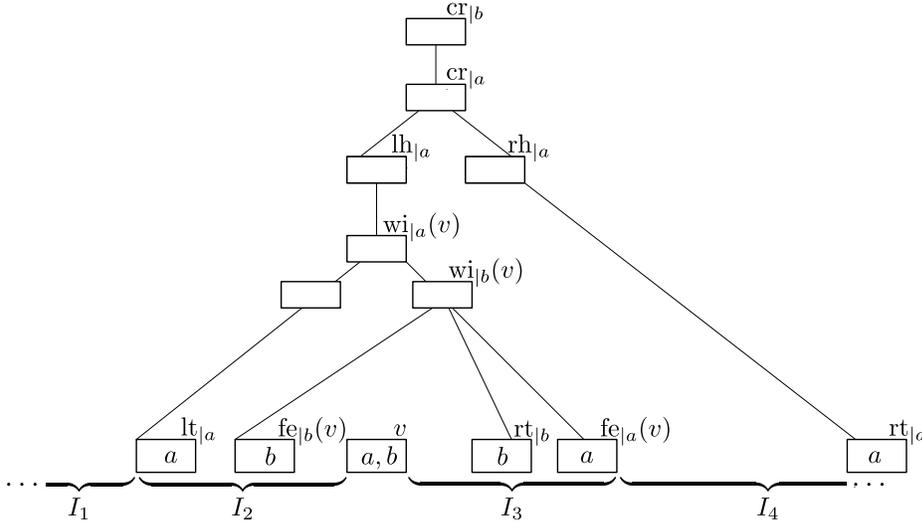


FIG. 10. In Case 2 interval  $I_4$  contains no  $bs$ . Contrary to the depiction,  $rt|_b$  and  $fe|_a(v)$  are not necessarily distinct.

Case 2. The wingtip  $rt|_b$  must lie in  $I_3$ , so  $v$  and  $rt|_b$  are both descendants of  $qu|_a(v)$ , the right child of  $wi|_a(v)$ . It follows that  $v$  is a hawk in  $\mathcal{T}|_b^*$  and that no descendants of  $wi|_b(v)$  are in interval  $I_1$ . Since  $fe|_b(v)$  is the leftmost descendant of  $wi|_b(v)$  in  $\mathcal{T}|_b^*$ , and  $fe|_b(v) \neq v$ , the distinct nodes  $lt|_a, fe|_b(v), v, rt|_b, rt|_a$  certify that  $a, b$  are double-nested in  $\mathcal{B}(v)$ . See Figure 10.  $\square$

Recurrence 7.3 is essentially the same as the correspond recurrence from [17].

RECURRENCE 7.3. *Let  $S$  be an  $m$ -block, order- $s$  DS sequence over an  $n$ -letter alphabet and  $\mathcal{T} = \mathcal{T}^*(S)$  be its canonical derivation tree. Define  $\Phi_s(n, m)$  to be the maximum number of feathers of one type (dove or hawk) in such a sequence, where feather is with respect to  $\mathcal{T}$ . For any  $s \geq 2$ ,*

$$\begin{aligned}\Phi_s(n, 2) &= 0 \\ \Phi_2(n, m) &< m\end{aligned}$$

and for any uniform block partition  $\{m_q\}_{1 \leq q \leq \hat{m}}$  and alphabet partition  $\{\hat{n}\} \cup \{\hat{n}_q\}_{1 \leq q \leq \hat{m}}$ ,

$$\Phi_s(n, m) \leq \sum_{q=1}^{\hat{m}} \Phi_s(\hat{n}_q, m_q) + \Phi_s(\hat{n}, \hat{m}) + \Phi_{s-1}(\hat{n}, m) + \hat{n}$$

*Proof.* Suppose we only wish to bound dove feathers. If there are only two blocks then all occurrences are wingtips and feathers are not wingtips. This gives the first equality. In the most extreme case every non-wingtip is a dove feather, so  $\Phi_s(n, m) \leq \lambda_s(n, m) - 2n$ . In particular,  $\Phi_2(n, m) \leq \lambda_2(n, m) - 2n < m$ . Decompose  $S$  into  $\hat{S}, \hat{S}', \hat{S}_q, \hat{S}_q', \hat{S}_q$  in the usual way with respect to the given uniform block partition. Let  $\hat{\mathcal{T}} = \mathcal{T}^*(\hat{S}')$  be the canonical derivation tree of the contracted global sequence  $\hat{S}'$ . It follows that  $\hat{S}_q$  is an order- $(s-1)$  DS sequence. Define  $\hat{\mathcal{T}}_q = \mathcal{T}^*(\hat{S}_q)$  to be its canonical derivation tree. The branching nodes on the left wing of  $\mathcal{T}_a$ , where  $a \in \Sigma(\hat{S}_q)$ , consist of (i) the branching nodes on the left wing of  $\hat{\mathcal{T}}_a$ , (ii) the branching nodes on the left wing of  $(\hat{\mathcal{T}}_q)_{|a}$ , and (iii) the crown  $\text{cr}_{|a}$  of  $(\hat{\mathcal{T}}_q)_{|a}$ , which is on the left wing of  $\mathcal{T}_a$  but not  $(\hat{\mathcal{T}}_q)_{|a}$ . Each branching node is identified with one feather in  $\mathcal{T}_a$ . The total number of branching nodes/feathers covered by (i), summed over all  $a \in \Sigma(\hat{S})$ , is at most  $\Phi_s(\hat{n}, \hat{m})$ . The total number covered by (ii), summed over all  $q \leq \hat{m}$  and  $a \in \Sigma(\hat{S}_q)$ , is  $\sum_q \Phi_{s-1}(\hat{n}_q, m_q) \leq \Phi_{s-1}(\hat{n}, m)$ . The number covered by (iii) is clearly  $\hat{n}$ , which gives the last inequality.  $\square$

Recurrence 7.4 generalizes [16, Rec. 3.1] and [17, Recs. 3.3 and 5.2], from DS sequences to double DS sequences. When  $s = 3$  or  $s \geq 4$  is even, Recurrence 7.4 is substantively no different than Recurrence 4.3 for  $\text{dblForm}(r, s+1)$ -free sequences.

RECURRENCE 7.4. *Let  $s, n$ , and  $m$  be the order, alphabet size, and block count parameters. Let  $\{m_q\}_{1 \leq q \leq \hat{m}}$  be a uniform block partition, where  $\hat{m} \geq 2$ , and  $\{\hat{n}\} \cup \{\hat{n}_q\}_{1 \leq q \leq \hat{m}}$  be an alphabet partition. When  $\hat{m} = 2$ , for any  $s \geq 3$ ,*

$$\lambda_s^{\text{dbl}}(n, m) \leq \sum_{q \in \{1, 2\}} \lambda_s^{\text{dbl}}(\hat{n}_q, m_q) + \lambda_{s-1}^{\text{dbl}}(2\hat{n}, m) + 2\hat{n}.$$

When  $\hat{m} > 2$  and either  $s = 3$  or  $s \geq 4$  is even,

$$\begin{aligned}\lambda_s^{\text{dbl}}(n, m) &\leq \sum_q \lambda_s^{\text{dbl}}(\hat{n}_q, m_q) + \lambda_s^{\text{dbl}}(\hat{n}, \hat{m}) + 2 \cdot \lambda_{s-1}^{\text{dbl}}(\hat{n}, m) + \lambda_{s-2}^{\text{dbl}}(\lambda_s(\hat{n}, \hat{m}), m) \\ &\quad + 2 \cdot \lambda_s(\hat{n}, \hat{m}),\end{aligned}$$

and when  $s \geq 5$  is odd,

$$\begin{aligned} \lambda_s^{\text{dbl}}(n, m) \leq & \sum_{q=1}^{\hat{m}} \lambda_s^{\text{dbl}}(\hat{n}_q, m_q) + \lambda_s^{\text{dbl}}(\hat{n}, \hat{m}) + 2 \cdot \lambda_{s-1}^{\text{dbl}}(\hat{n}, m) + \lambda_{s-2}^{\text{dbl}}(2 \cdot \Phi_s(\hat{n}, \hat{m}), m) \\ & + 4 \cdot \Phi_s(\hat{n}, \hat{m}) + \lambda_{s-3}^{\text{dbl}}(\lambda_s(\hat{n}, \hat{m}), m) + 2 \cdot \lambda_s(\hat{n}, \hat{m}) \end{aligned}$$

*Proof.* First consider the case when  $s \geq 5$  is odd. Let  $S$  be an order- $s$  double DS sequence, decomposed into  $\hat{S}$  and  $\{\check{S}_q\}$  as usual. The contribution of local symbols is  $\sum_q \lambda_s^{\text{dbl}}(\hat{n}_q, m_q)$ . If a global symbol occurs exactly once in an  $\hat{S}_q$  this occurrence is a *singleton*. Let  $\dot{S} < \hat{S}$  be the subsequence of singletons and  $\ddot{S} < \hat{S}$  be the subsequence of non-singletons. By definition  $\dot{S}$  is partitioned into  $\hat{m}$  blocks, so  $|\dot{S}| \leq \lambda_s^{\text{dbl}}(\hat{n}, \hat{m})$ . Symbols in  $\Sigma(\check{S}_q)$  are classified as *first*, *last*, and *middle* if they appear, in  $\dot{S}$ , after  $\check{S}_q$  but not before, before  $\check{S}_q$  but not after, and both before and after  $\check{S}_q$ , respectively. In the worst case these three criteria are exhaustive. However, it may be that all non-singleton occurrences of a symbol appear *exclusively* in  $\Sigma(\check{S}_q)$ . In this case we call the symbol *first* if it appears after interval  $q$  in  $\dot{S}$  and *last* if it is not first and appears before interval  $q$  in  $\dot{S}$ . Define  $\acute{S}_q, \check{S}_q, \bar{S}_q < \ddot{S}_q$  to be the subsequences of first, last, and middle occurrences in  $\check{S}_q$ .

If we remove the last occurrence of each letter from  $\acute{S}_q$ , or the first occurrence of each letter from  $\check{S}_q$ , the resulting sequence is an order- $(s-1)$  double DS sequence. The contribution of first and last non-singletons is therefore at most

$$\sum_q \left[ \lambda_{s-1}^{\text{dbl}}(\acute{n}_q, m_q) + \acute{n}_q + \lambda_{s-1}^{\text{dbl}}(\check{n}_q, m_q) + \check{n}_q \right] \leq 2(\lambda_{s-1}^{\text{dbl}}(\hat{n}, m) + \hat{n}).$$

Obtain  $\check{S}' = B_1 \cdots B_{\hat{m}}$  from  $\check{S}$  by contracting each interval  $\check{S}_q$  into a single block  $B_q$ . Since occurrences in  $\check{S}'$  each represent at least two occurrences in  $\check{S}$ , we can conclude<sup>15</sup> that  $|\check{S}'| \leq \lambda_s(\hat{n}, \hat{m})$ .

Let  $\check{T} = \mathcal{T}^*(\check{S}')$  be the canonical derivation tree of  $\check{S}'$ . Define  $\tilde{S}'$  to be the subsequence of  $\check{S}'$  consisting of feathers with respect to  $\check{T}$  (both dove and hawk) and let  $\tilde{S}$  be the subsequence of  $\check{S}$  beget by symbols in  $\tilde{S}'$ . It follows that  $|\tilde{S}'| \leq 2 \cdot \Phi_s(\hat{n}, \hat{m})$  since  $\Phi_s$  only counts feathers of one type (dove or hawk). Define  $\dot{S}' < \check{S}'$  to be the subsequence of non-feather, non-wingtips with respect to  $\check{T}$ , and define  $\dot{S} < \check{S}$  analogously. Since  $\tilde{S}$  consists solely of middle symbols, removing the first and last occurrence of each letter in  $\tilde{S}_q$  leaves an order- $(s-2)$  double DS sequence, hence

$$\begin{aligned} |\tilde{S}| &= \sum_q |\tilde{S}_q| \leq \sum_q (\lambda_{s-2}^{\text{dbl}}(\tilde{n}_q, m_q) + 2\tilde{n}_q) \\ &\leq \lambda_{s-2}^{\text{dbl}} \left( \sum_q \tilde{n}_q, m \right) + 2 \sum_q \tilde{n}_q \\ &\leq \lambda_{s-2}^{\text{dbl}}(|\tilde{S}'|, m) + 2(|\tilde{S}'|) \\ &\leq \lambda_{s-2}^{\text{dbl}}(2 \cdot \Phi_s(\hat{n}, \hat{m}), m) + 4 \cdot \Phi_s(\hat{n}, \hat{m}) \end{aligned}$$

<sup>15</sup>This is not quite true. As discussed in Remark 4.2, we can make this inference when bounding  $\lambda_s^{\text{dbl}}$  asymptotically.

We have accounted for every part of  $S$  except for  $\mathring{S}$ . Fix an interval  $q$  and  $a, b \in \Sigma(\mathring{S}_q)$ . Since  $a, b \in B_q$  are neither feathers nor wingtips in  $\mathring{T}$ , Lemma 7.2 implies that  $\mathring{S}$  contains  $\underline{a} b b \mathring{S}_q b b \underline{a}$ . Suppose we remove the first and last occurrence of each letter in  $\mathring{S}_q$ . (These letters are underlined below.) The resulting sequence must be an order- $(s-3)$  double DS sequence, for if it contained a doubled alternating sequence with length  $s-1$ , which is even, we would see either

$$a b b \left| \overbrace{\underline{a} a b b \cdots a a b \underline{b}}^{s-1 \text{ alternations}} \right| b b a \quad \text{or} \quad a b b \left| \overbrace{\underline{b} b a a \cdots b b a \underline{a}}^{s-1 \text{ alternations}} \right| b b a,$$

contradicting the fact that  $S$  is an order- $s$  double DS sequence. We can therefore bound  $|\mathring{S}|$  by

$$\begin{aligned} \sum_q |\mathring{S}_q| &\leq \sum_q (\lambda_{s-3}^{\text{dbl}}(\mathring{n}_q, m_q) + 2\mathring{n}_q) \\ &\leq \lambda_{s-3}^{\text{dbl}} \left( \sum_q \mathring{n}_q, m \right) + 2 \sum_q \mathring{n}_q \\ &\leq \lambda_{s-3}^{\text{dbl}}(|\mathring{S}'|, m) + 2|\mathring{S}'| \\ &\leq \lambda_{s-3}^{\text{dbl}}(|\mathring{S}'| - 2\hat{n}, m) + 2(|\mathring{S}'| - 2\hat{n}) \\ &\leq \lambda_{s-3}^{\text{dbl}}(\lambda_s(\hat{n}, \hat{m}) - 2\hat{n}, m) + 2(\lambda_s(\hat{n}, \hat{m}) - 2\hat{n}) \end{aligned}$$

This establishes the recurrence for odd  $s \geq 5$ . When  $s = 3$  or  $s \geq 4$  is even, we ignore the distinction between feathers and non-feathers and bound  $|\mathring{S}|$  by  $\lambda_{s-2}^{\text{dbl}}(\lambda_s(\hat{n}, \hat{m}) - 2\hat{n}, m) + 2(\lambda_s(\hat{n}, \hat{m}) - 2\hat{n})$ . When  $S = S_1 S_2$  consists of  $\hat{m} = 2$  intervals, no symbols are classified as middle, so it suffices to account for first, last, and local occurrences only. After discarding the last occurrence of each symbol from  $\mathring{S}_1$  and the first from  $\mathring{S}_2$ , what remains are order- $(s-1)$  double DS sequences, so  $|\mathring{S}| \leq 2\hat{n} + \lambda_{s-1}^{\text{dbl}}(\hat{n}, m_1) + \lambda_{s-1}^{\text{dbl}}(\hat{n}, m_2) \leq 2\hat{n} + \lambda_{s-1}^{\text{dbl}}(2\hat{n}, m)$ .  $\square$

Recurrence 7.5 combines the content of [17, Recs. 3.3 and 5.2] but is presented in the style of Recurrence 7.4. The proof is essentially the same as that of Recurrence 7.4 except that we do not need to distinguish singletons from non-singletons, nor do we need to remove symbols from  $\mathring{S}_q, \mathring{S}_q, \mathring{S}_q, \mathring{S}_q$ , or  $\mathring{S}_q$  in order to make them double DS sequences with order  $s-1$  or  $s-2$  or  $s-3$ , as the case may be.

**RECURRENCE 7.5.** *Let  $s, n$ , and  $m$  be the order, alphabet size, and block count parameters. Let  $\{m_q\}_{1 \leq q \leq \hat{m}}$  be a uniform block partition, where  $\hat{m} \geq 2$ , and  $\{\hat{n}\} \cup \{\mathring{n}_q\}_{1 \leq q \leq \hat{m}}$  be an alphabet partition. When  $\hat{m} = 2$ , for any  $s \geq 3$ ,*

$$\lambda_s(n, m) \leq \sum_{q \in \{1, 2\}} \lambda_s(\mathring{n}_q, m_q) + \lambda_{s-1}(2\hat{n}, m).$$

When  $\hat{m} > 2$  and either  $s = 3$  or  $s \geq 4$  is even,

$$\lambda_s(n, m) \leq \sum_q \lambda_s(\mathring{n}_q, m_q) + 2 \cdot \lambda_{s-1}(\hat{n}, m) + \lambda_{s-2}(\lambda_s(\hat{n}, \hat{m}) - 2\hat{n}, m)$$

and when  $s \geq 5$  is odd,

$$\begin{aligned} \lambda_s(n, m) \leq & \sum_{q=1}^{\hat{m}} \lambda_s(\tilde{n}_q, m_q) + 2 \cdot \lambda_{s-1}(\hat{n}, m) + \lambda_{s-2}(2 \cdot \Phi_s(\hat{n}, \hat{m}), m) \\ & + \lambda_{s-3}(\lambda_s(\hat{n}, \hat{m}), m) \end{aligned}$$

Lemma 7.6 states some bounds on  $\Phi_s$ ,  $\lambda_s$ , and  $\lambda_s^{\text{dbl}}$  in terms of coefficients  $\{\phi_{s,i}, \delta_{s,i}, \delta_{s,i}^{\text{dbl}}\}$  and the  $i$ th row-inverse of Ackermann's function, for any  $i \geq 1$ . Refer to [17, Appendix B] for proofs of similar lemmas, and to the discussion following Lemma 4.4.

LEMMA 7.6. *Fix parameters  $i \geq 1$ ,  $s \geq 3$ , and  $c \geq s - 2$  and let  $n, m$  be the alphabet size and block count. Let  $j$  be minimal such that  $m \leq (a_{i,j})^c$ . Then  $\Phi_s, \lambda_s$ , and  $\lambda_s^{\text{dbl}}$  are bounded by*

$$\begin{aligned} \Phi_s(n, m) &\leq \phi_{s,i}(n + O((cj)^{s-2}m)) \\ \lambda_s(n, m) &\leq \delta_{s,i}(n + O((cj)^{s-2}m)) \\ \lambda_s^{\text{dbl}}(n, m) &\leq \delta_{s,i}^{\text{dbl}}(n + O((cj)^{s-2}m)) \end{aligned}$$

where  $\{\phi_{s,i}, \delta_{s,i}, \delta_{s,i}^{\text{dbl}}\}$  are defined as follows.

$$\begin{aligned} \phi_{2,i} &= 0 && \text{all } i \\ \phi_{s,1} &= \phi_{s-1,1} + 1 && s \geq 3 \\ \phi_{s,i} &= \phi_{s,i-1} + \phi_{s-1,i} + 1 && s \geq 3, i \geq 2 \\ \delta_{1,i} &= 1 && \text{all } i \\ \delta_{2,i} &= 2 && \text{all } i \\ \delta_{1,i}^{\text{dbl}} &= 2 && \text{all } i \\ \delta_{2,i}^{\text{dbl}} &= 5 && \text{all } i \\ \delta_{s,1} &= 2\delta_{s-1,1} = 2^{s-1} && s \geq 3 \\ \delta_{s,1}^{\text{dbl}} &= 2(\delta_{s-1,1}^{\text{dbl}} + 1) = 2^{s+1} - 2^{s-2} - 2 && s \geq 3 \\ \delta_{s,i} &= \begin{cases} 2\delta_{s-1,i} + \delta_{s-2,i}(\delta_{s,i-1} - 2) & s = 3 \text{ or even } s \geq 4 \\ 2\delta_{s-1,i} + 2\delta_{s-2,i}\phi_{s,i-1} + \delta_{s-3,i}\delta_{s,i-1} & \text{odd } s \geq 5 \end{cases} \\ \delta_{s,i}^{\text{dbl}} &= \begin{cases} \delta_{s,i-1}^{\text{dbl}} + 2\delta_{s-1,i}^{\text{dbl}} + (\delta_{s-2,i}^{\text{dbl}} + 2)\delta_{s,i-1} & s = 3 \text{ or even } s \geq 4 \\ \delta_{s,i-1}^{\text{dbl}} + 2\delta_{s-1,i}^{\text{dbl}} + 2(\delta_{s-2,i}^{\text{dbl}} + 2)\phi_{s,i-1} + (\delta_{s-3,i}^{\text{dbl}} + 2)\delta_{s,i-1} & \text{odd } s \geq 5 \end{cases} \end{aligned}$$

When applying Lemma 7.6, the tightest bounds are obtained by setting  $i = \alpha(n, m) + O(1)$ , which is  $\alpha(n) + O(1)$  whenever  $j = O(1)$ . Lemma 7.7 gives closed form bounds on the coefficients  $\{\delta_{s,i}, \delta_{s,i}^{\text{dbl}}, \phi_{s,i}\}$ , which immediately yield sharp bounds on the extremal functions  $\lambda_s(n, m)$  and  $\lambda_s^{\text{dbl}}(n, m)$  for DS and double DS sequences

partitioned into blocks.

LEMMA 7.7. *For all  $s \geq 3, i \geq 1$ , we have*

$$\begin{aligned}\phi_{s,i} &= \binom{i+s-2}{s-2} - 1 \\ \delta_{3,i} &= 2i + 2 \\ \delta_{3,i}^{\text{dbl}} &= \Theta(i^2) \\ \delta_{4,i}, \delta_{4,i}^{\text{dbl}} &= \Theta(2^i) \\ \delta_{5,i}, \delta_{5,i}^{\text{dbl}} &= \Theta(i2^i) \\ \delta_{s,i}, \delta_{s,i}^{\text{dbl}} &\leq 2^{\binom{i+O(1)}{t}}\end{aligned}\quad \text{where } t = \lfloor \frac{s-2}{2} \rfloor.$$

*Proof.* The expression for  $\phi_{s,i}$  holds in the base cases, when  $s = 2$  or  $i = 1$ . By Pascal's identity it holds in general since

$$\phi_{s,i} = \phi_{s,i-1} + \phi_{s-1,i} + 1 = \binom{i+s-3}{s-2} + \binom{i+s-3}{s-3} - 1 = \binom{i+s-2}{s-2} - 1.$$

When  $s \in \{3, 4\}$ ,  $\delta_{s,i}$  and  $\delta_{s,i}^{\text{dbl}}$  are identical to  $\pi_{s,i}$  and  $\pi_{s,i}^{\text{dbl}}$ , and therefore satisfy the same bounds from Lemma 4.5. Define  $C_4$  such that  $\delta_{4,i} \leq 2^{i+C_4}$ . Assuming inductively that for some sufficiently large  $C_5$ ,  $\delta_{5,i-1} \leq (i-1)2^{(i-1)+C_5}$ , we have

$$\begin{aligned}\delta_{5,i} &\leq 2\delta_{4,i} + 2\delta_{3,i}\phi_{5,i-1} + \delta_{2,i}\delta_{5,i-1} \\ &\leq 2^{i+C_4+1} + 2(2i+2) \cdot \binom{i+2}{3} + 2 \cdot (i-1)2^{i-1+C_5} \\ &\leq i2^{i+C_5}.\end{aligned}$$

We claim that there are constants  $\{C_s\}$  such that, for all  $s > 5$ ,  $\delta_{s,i} \leq 2^{\binom{i+C_s}{t}}$ . When  $s > 4$  is even,

$$\begin{aligned}\delta_{s,i} &\leq 2\delta_{s-1,i} + \delta_{s-2,i}\delta_{s,i-1} \\ &\leq 2^{\binom{i+C_{s-1}}{t-1}+1} + 2^{\binom{i+C_{s-2}}{t-1}}2^{\binom{i-1+C_s}{t}} \\ &\leq 2^{\binom{i+C_s}{t}}, \text{ for some } C_s > C_{s-1} > C_{s-2}.\end{aligned}$$

When  $s > 5$  is odd, whether  $s-2 = 5$  or not,  $\delta_{s-2,i} \leq i2^{\binom{i+C_{s-2}}{t-1}}$  by the inductive hypothesis, so

$$\begin{aligned}\delta_{s,i} &\leq 2\delta_{s-1,i} + 2\delta_{s-2,i}\phi_{s,i-1} + \delta_{s-3,i}\delta_{s,i-1} \\ &\leq 2^{\binom{i+C_{s-1}}{t}+1} + i2^{\binom{i+C_{s-2}}{t-1}+1} \cdot \binom{i+s-3}{s-2} + 2^{\binom{i+C_{s-3}}{t-1}}2^{\binom{i-1+C_s}{t}} \\ &\leq 2^{\binom{i+C_{s-1}}{t}+1} + i2^{\binom{i+C_{s-2}}{t-1}+1} \cdot \binom{i+s-3}{s-2} + 2^{-(C_s-C_{s-3})}2^{\binom{i+C_s}{t-1}+\binom{i-1+C_s}{t}}\end{aligned}\quad (7.1)$$

$$\leq 2^{\binom{i+C_s}{t}}.\quad (7.2)$$

Inequality (7.1) follows since  $t-1 \geq 1$  and Inequality (7.2) follows since, for  $C_s$  sufficiently large,  $2^{\binom{i+C_s}{t}}$  dominates both  $\text{poly}(i) \cdot 2^{\binom{i+C_{s-2}}{t-1}}$  and  $2^{\binom{i+C_{s-1}}{t}+1}$ . It is

straightforward to show the same bounds hold on  $\delta_{s,i}^{\text{dbl}}$ , for  $s \geq 4$ , with respect to different constants  $\{D_s\}$ . That is,  $\delta_{s,i}^{\text{dbl}} \leq 2^{\binom{i+D_s}{t}}$  when  $s \neq 5$  and  $\delta_{5,i}^{\text{dbl}} \leq i2^{i+D_5}$ .  $\square$

Choosing  $i = \alpha(n, m) + O(1)$ , Lemmas 7.6 and 7.7 imply that

$$\begin{aligned}\lambda_3(n, m) &= O((n+m)\alpha(n, m)) \\ \lambda_3^{\text{dbl}}(n, m) &= O((n+m)\alpha^2(n, m)) \\ \lambda_4(n, m), \lambda_4^{\text{dbl}}(n, m) &= O((n+m)2^{\alpha(n, m)}) \\ \lambda_5(n, m), \lambda_5^{\text{dbl}}(n, m) &= O((n+m)\alpha(n, m)2^{\alpha(n, m)}) \\ \lambda_s(n, m), \lambda_s^{\text{dbl}}(n, m) &= O((n+m)2^{\alpha^t(n, m)/t! + O(\alpha^{t-1}(n, m))})\end{aligned}$$

When  $m = O(n)$  these bounds are all sharp, with the exception of  $\lambda_3^{\text{dbl}}$ , which was already handled in Section 6. Using the best transformations from 2-sparse to blocked sequences from Lemma 3.1, we obtain all the bounds on  $\lambda_s$  and  $\lambda_s^{\text{dbl}}$  claimed in Theorem 1.3, except at  $s = 5$ , where we only get  $\lambda_5(n) = O(\alpha(\alpha(n))) \cdot \lambda_5(n, 3n)$  and  $\lambda_5^{\text{dbl}}(n) = O(\alpha(\alpha(n))) \cdot \lambda_5^{\text{dbl}}(n, 3n)$ . Refer to [17, §6.2] for an ad hoc method to eliminate this  $\alpha(\alpha(n))$  factor.

**8. Generalized Constructions of Nonlinear Sequences.** Lower bounds on generalized DS sequences are generally expressed in an ad hoc manner. Nonetheless, all prior constructions can be expressed in terms of three basic operations: *composition*, *preshuffling*, and *postshuffling*.<sup>16</sup> The nominal purpose of this section is to establish specific lower bounds on certain forbidden subsequences. However, its true contribution is a new succinct notation that is expressive enough to capture prior sequence constructions *and* suggest numerous variations.

Recall from Section 2.1 that the difference between postshuffling and preshuffling is in how blocks of one sequence are merged with copies of another. In  $U_{\text{sub}} \otimes U_{\text{bot}}$  symbols from  $U_{\text{sub}}$  are inserted at the *end* of blocks in copies of  $U_{\text{bot}}$  whereas in  $U_{\text{sub}} \oplus U_{\text{bot}}$  they are inserted at the beginning of blocks. It is not immediately clear why these two shuffling strategies should yield sequences with different properties. Consider the projection of symbols  $R = \{a, \dots, z\}$  in a common block  $B$  of  $U_{\text{top}}$ , where all symbols in  $R$  are middle occurrences in  $B$ . If  $U_{\text{top}}$  was constructed via a series of composition and *postshuffling* operations, the projection of  $U_{\text{top}}$  onto  $R$ , ignoring repetitions, would be

$$a b \cdots z (z y \cdots a) z y \cdots a,$$

whereas if *preshuffling* were used the projection onto  $R$  would be

$$a b \cdots z (a b \cdots z) z y \cdots a.$$

In a subsequent composition event  $U_{\text{sub}} = U_{\text{top}} \circ U_{\text{mid}}$ , the canonical ordering of  $R$  in  $U_{\text{mid}}(B)$  is identical to their ordering in  $U_{\text{top}}$ , in the case of preshuffling, or the reversal of that ordering in the case of postshuffling.

In this section we explore the complexity of sequences avoiding “zig-zagging” patterns, which can be viewed as one natural generalization of Davenport–Schinzel

<sup>16</sup>The one possible exception to this blanket statement is Nivasch’s construction [16] of order-3 DS sequences with length  $2n\alpha(n) - O(n)$ . That construction’s shuffling operation selectively applies postshuffling to first occurrences and preshuffling to last occurrences, so it is still possible to view it through the prism of these three basic operations.

sequences. Recall the definitions of  $N_k$ ,  $M_k$ , and  $Z_k$ .

$$\begin{aligned} N_k &= 12 \cdots (k+1)k \cdots 12 \cdots (k+1) \\ M_k &= 12 \cdots (k+1)k \cdots 12 \cdots (k+1)k \cdots 1 \\ Z_k &= 12 \cdots (k+1)k \cdots 12 \cdots (k+1)k \cdots 12 \cdots (k+1) \end{aligned}$$

Note that  $N_1 = abab$ ,  $M_1 = ababa$ , and  $Z_1 = ababab$  generalize order-2, -3, and -4 Davenport–Schinzel sequences. Klazar and Valtr [14] and Pettie [21] proved that  $\text{Ex}(N_k, n) = \Theta(\lambda_2(n)) = \Theta(n)$  and that for any  $k \geq 1$ ,  $\text{Ex}(\{M_k, ababab\}, n) = \Theta(\lambda_3(n)) = \Theta(n\alpha(n))$ . (That is, avoiding *both*  $M_k$  and  $ababab$  are equivalent to just avoiding  $M_1$ .) One might guess that zig-zagging patterns, in general, mimic the behavior of the corresponding order- $s$  DS sequences.

We prove two results that, taken together, are rather surprising. Theorems 8.5 and 8.6 state the following in a more precise fashion.

- (1) For all  $t$ , there exists a  $k$  such that  $\text{Ex}(M_k, n) = \Omega(n\alpha^t(n))$ .
- (2) For all  $t$ , there exists a  $k$  such that  $\text{Ex}(Z_k, n) = \Omega(n2^{(1+o(1))\alpha^t(n)/t!})$ .

*Overview.* We define two classes of non-linear sequences. Class I sequences have lengths  $\Theta(n\alpha^t(n))$  and Class II sequences have length  $n2^{(1+o(1))\alpha^t(n)/t!}$ , for any  $t \geq 1$ . Both Class I and Class II sequences are parameterized by a binary *pattern*  $\pi$  over the alphabet  $\{/, \backslash\}$ , that is,  $\pi = \pi_1\pi_2 \cdots \pi_{|\pi|} \in \{/, \backslash\}^{|\pi|}$ . The diagonals in  $\pi$  have the following interpretation. Consider any set  $\{a_1, \dots, a_l\}$  of symbols in a sequence  $T_\pi$  of type  $\pi$ . A *maximally* intertwined configuration is one in which each pair of symbols in  $\{a_1, \dots, a_l\}$  alternate the maximum number of times. In  $T_\pi$  all maximally intertwined configurations will take the form  $A^{\pi_1}A^{\pi_2} \cdots A^{\pi_{|\pi|}}$ , where  $A^/ = a_1 \cdots a_l$  and  $A^\backslash = a_l \cdots a_1$ . Class I and II sequences are defined in Sections 8.1 and 8.2 and their forbidden sequences analyzed in Section 8.3.

**8.1. Class I Sequences.** The sequence  $T_\pi(i, j)$  consists of a mixture of live and dead blocks. It is parameterized by a pattern  $\pi$ , which always begins with  $/$ . The base cases for  $T_\pi$  are given below. (Recall that live blocks are indicated with parentheses and dead blocks with angular brackets.)

$$\begin{aligned} T_{\wedge}(i, j) &= (12 \cdots j) \langle j \cdots 21 \rangle && \text{one live block, one dead, for any } i \\ T_{\swarrow}(i, j) &= (12 \cdots j) \langle 12 \cdots j \rangle && \text{one live block, one dead, for any } i \\ T_\pi(1, j) &= \begin{cases} (12 \cdots j) \langle j \cdots 21 \rangle & \text{if } \pi_{|\pi|} = \backslash \text{ and } |\pi| > 2 \\ (12 \cdots j) \langle 12 \cdots j \rangle & \text{if } \pi_{|\pi|} = / \text{ and } |\pi| > 2 \end{cases} \\ T_\pi(i, 0) &= ()^2 && \text{two empty live blocks, any } \pi \end{aligned}$$

Note that  $T_\pi(1, j)$  is identical to either  $T_{\wedge}(\cdot, j)$  or  $T_{\swarrow}(\cdot, j)$ , depending on the last character of  $\pi$ . For the inductive case, when  $i > 1, j > 0$ , and  $|\pi| > 2$ ,

$$T_\pi(i, j) = \begin{cases} T_{\text{sub}} \otimes T_{\text{bot}} = (T_{\text{top}} \circ T_{\text{mid}}) \otimes T_{\text{bot}} & \text{if } \pi_{|\pi|} = \setminus \\ T_{\text{sub}} \otimes T_{\text{bot}} = (T_{\text{top}} \circ T_{\text{mid}}) \otimes T_{\text{bot}} & \text{if } \pi_{|\pi|} = / \end{cases}$$

where  $T_{\text{bot}} = T_\pi(i, j - 1)$

$$T_{\text{mid}} = T_{\pi^-}(i, \lVert T_{\text{bot}} \rVert) \quad \pi^- = \pi_1 \cdots \pi_{|\pi|-1}$$

$$T_{\text{top}} = T_\pi(i - 1, \lVert T_{\text{mid}} \rVert)$$

The following facts can easily be proved about  $T_\pi(i, j)$  by induction.

1. The first occurrence of every symbol appears in a live block and live blocks consist solely of first occurrences.
2. All live blocks have length exactly  $j$ . The length of dead blocks varies, as does the number of dead blocks between consecutive live blocks.
3. Each symbol occurs with the same multiplicity,  $\nu_{\pi, i}$ , defined below. Hence  $\lVert T \rVert = \nu_{\pi, i} \lVert T \rVert = \nu_{\pi, i} \cdot j \cdot \lVert T \rVert$ .

The construction of  $T_\pi$  gives us an inductive expression for the multiplicity  $\nu_{\pi, i}$  of symbols in  $T_\pi(i, j)$ .

$$\begin{aligned} \nu_{\pi, i} &= 2 && \text{for } |\pi| = 2 \text{ and all } i \\ \nu_{\pi, 1} &= 2 && \text{for all } \pi \\ \nu_{\pi, i} &= \nu_{\pi, i-1} + \nu_{\pi^-, i} - 1 && \text{where } \pi^- = \pi_1 \cdots \pi_{|\pi|-1} \end{aligned}$$

A short proof by induction shows that  $\nu_{\pi, i}$  has the closed form

$$\nu_{\pi, i} = \binom{i + |\pi| - 3}{|\pi| - 2} + 1 \quad \text{for all } i \geq 1, |\pi| \geq 2$$

It can be shown that  $i = \alpha(n, m) + O(1)$ , where  $n = \lVert T_\pi(i, j) \rVert$  and  $m = \lVert \lVert T_\pi(i, j) \rVert \rVert$ , from which it follows that  $T_\pi(i, j)$  has length  $\Theta(n\alpha^{|\pi|-2}(n, m))$ , and length  $\Theta(n\alpha^{|\pi|-2}(n))$  if  $j = O(1)$ . Theorem 8.1 summarizes two results from [9, 20, 22] using the  $T_\pi$  notation.

THEOREM 8.1. ([9, 20, 22])

1.  $ababa, abcacbc \not\prec T_{/\setminus}$ .
2.  $abaaba, abcacbc \not\prec T_{/\wedge}$ .

As a consequence both  $\text{Ex}(ababa, n)$  and  $\text{Ex}(abcacbc, n)$  are  $\Omega(n\alpha(n))$ , which is asymptotically tight.

**8.2. Class II Sequences.** Class II Sequences consist *solely* of live blocks. They are parameterized by binary patterns, which are restricted to being even-length palindromes, starting with  $/$  and ending with  $\setminus$ . If  $\pi = \pi_1 \cdots \pi_{|\pi|}$ , its *flip*  $\text{flip}(\pi)$  is obtained by flipping the direction of each diagonal and its *truncation*  $\pi^-$  is obtained by trimming  $\pi_1$  and  $\pi_{|\pi|}$ . For example, if  $\pi = / \setminus \setminus \setminus / / \setminus$ ,  $\text{flip}(\pi^-) = / / \setminus \setminus \setminus$ .

The base cases for  $U_\pi$  are given below. The sequence  $U_\pi(i, j)$  has the property that each block has length  $j$  and each symbol has multiplicity  $\mu_{\pi, i}$ , which will be

defined below.

$$\begin{array}{ll}
U_{\wedge}(i, j) = (12 \cdots j)(j \cdots 21) & \text{two blocks, for any } i \\
U_{\pi}(1, j) = (12 \cdots j)(j \cdots 21) & \text{two blocks, for any } \pi \\
U_{\pi}(0, j) = (12 \cdots j) & \text{one block, for any } \pi \\
U_{\pi}(i, 1) = (1)^{\mu_{\pi, i}} & \mu_{\pi, i} \text{ identical blocks}
\end{array}$$

For the inductive case, when  $i > 1, j > 0$ , and  $|\pi| > 2$ , we have

$$U_{\pi}(i, j) = \begin{cases} U_{\text{sub}} \otimes U_{\text{bot}} = (U_{\text{top}} \circ T_{\text{mid}}) \otimes U_{\text{bot}} & \text{if } \pi_2 \pi_{|\pi|-1} = \wedge \\ U_{\text{sub}} \otimes U_{\text{bot}} = (U_{\text{top}} \circ T_{\text{mid}}) \otimes U_{\text{bot}} & \text{if } \pi_2 \pi_{|\pi|-1} = \searrow \end{cases}$$

where  $U_{\text{bot}} = U_{\pi}(i, j-1)$

$$U_{\text{mid}} = \begin{cases} U_{\pi^-}(i, \llbracket T_{\text{bot}} \rrbracket) & \text{if } \pi_2 \pi_{|\pi|-1} = \wedge \\ U_{\text{flip}(\pi^-)}(i, \llbracket T_{\text{bot}} \rrbracket) & \text{if } \pi_2 \pi_{|\pi|-1} = \searrow \end{cases}$$

$$U_{\text{top}} = U_{\pi}(i-1, \llbracket T_{\text{mid}} \rrbracket)$$

The construction of  $U_{\pi}$  is a strict generalization of the  $U_s$  sequences defined in Section 2, for even  $s$ . Note that when  $\pi = (\wedge \searrow)^{s/2}$ , only postshuffling is used, since  $\text{flip}(\pi^-) = (\wedge \searrow)^{s/2-1}$ . The multiplicity  $\mu_{\pi, i}$  of symbols in  $U_{\pi}(i, j)$  is not affected by which shuffling operation is used, so the analysis from Section 2 still holds:  $\mu_{\pi, i} = 2^{\binom{i+t-1}{i}} \geq 2^{i^t/t!}$ , where  $t = (|\pi| - 2)/2$ , and  $i = \alpha(\|U_{\pi}(i, j)\|, \llbracket U_{\pi}(i, j) \rrbracket) + O(1)$ .

**8.3. Analysis of  $T_{\pi}$  and  $U_{\pi}$ .** Lemmas 8.2 and 8.3 isolate some properties of  $T_{\pi}$  useful in the analysis of  $M$ -shaped sequences and comb-shaped sequences.

LEMMA 8.2. *Let  $T_{\text{sh}} = T_{\pi}(i, j)$ , where  $i$  and  $j$  are arbitrary. Let  $\chi = \pi_{|\pi|}$  and  $\chi' = \pi_{|\pi|-1}$  be the last and second to last characters of  $\pi$ , and let  $T_{\text{top}}, T_{\text{mid}}, T_{\text{sub}}$ , and  $T_{\text{bot}}$  be the sequences arising in the formation of  $T_{\text{sh}}$ .*

1. *If  $abba < T_{\text{sh}}$  or  $baba < T_{\text{sh}}$  then it cannot be that  $b \in \Sigma(T_{\text{sub}})$  while  $a \in \Sigma(T_{\text{bot}}^*)$ .*
2. *If  $a < b$  share a live block in one of  $T_{\text{top}}, T_{\text{bot}}$ , or  $T_{\text{sh}}$ , then this sequence's projection onto  $\{a, b\}$  has the form  $(ab)a^*b^*$  if  $\chi = /$  and  $(ab)b^*a^*$  if  $\chi = \setminus$ .*
3. *If  $a_1 < \cdots < a_l$  share a live block in  $T_{\text{sub}}$ , then its projection onto  $\{a_1, \dots, a_l\}$  has the form  $(a_1 \dots a_l)A^{\chi'}A^{\chi}$  where  $A^{\chi} = a_1^* \dots a_l^*$  and  $A^{\setminus} = a_l^* \dots a_1^*$ .*

LEMMA 8.3. *Whereas  $ababa \not\prec T_{\wedge \searrow}$ ,  $abaaba \not\prec T_{\pi}$ , for any pattern  $\pi \in \{/\wedge, \wedge \setminus, // \}$ .*

*Proof.* Lemma 8.2(1) implies that  $ababa$  cannot be introduced by a shuffling event, but must first appear in  $T_{\text{sub}} = T_{\text{top}} \circ T_{\text{mid}}$  from a composition event. Moreover,  $abaaba$  could not arise in  $T_{\text{sub}}$  from an occurrence of  $ababa$  in  $T_{\text{top}}$  since, in such an occurrence, the middle  $a$  would necessarily be in a dead block and could therefore not beget multiple  $a$ s in  $T_{\text{sub}}$ . It must be that  $a$  and  $b$  share a common live block in  $T_{\text{top}}$ , so its projection onto  $\{a, b\}$  is contained in  $(ab)a^*b^*$ , if  $\pi_3 = /$ , and  $(ab)b^*a^*$  if  $\pi_3 = \setminus$ . Since  $T_{\text{mid}}$  is either  $T_{//}$  or  $T_{\wedge \searrow}$ , the projection of  $T_{\text{sub}}$  onto  $\{a, b\}$  is one of

$$(ab) \langle ba \rangle a^*b^* \quad \text{or} \quad (ab) \langle ba \rangle b^*a^* \quad \text{or} \quad (ab) \langle ab \rangle a^*b^* \quad \text{or} \quad (ab) \langle ab \rangle b^*a^*.$$

The first is  $ababa$ -free while the remaining are  $abaaba$ -free.  $\square$

In Theorem 8.5 we prove that  $\text{Ex}(M_{2^k}, n) = \Omega(n\alpha^{k+1}(n))$  by induction. Lemma 8.4 handles the base case for  $M_2$ .

LEMMA 8.4.  $M_2 = abcbabcba \prec T_\pi$ , for any of the length-4 patterns  $\pi \in \prec\{/, \backslash\}^2\prec$ .

*Proof.* Since  $M_2$  contains a subsequence of the form  $xyxyx$  for each pair of symbols  $\{x, y\} \subset \{a, b, c\}$ , any instance of  $M_2$  must first arise in  $T_{\text{sub}} = T_{\text{top}} \circ T_{\text{mid}}$  from a composition event, not in  $T_{\text{sh}} = T_{\text{sub}} \otimes T_{\text{mid}}$  from a shuffling event. Here  $T_{\text{mid}}$  is defined by any of the four patterns  $\pi^- \in \prec\{/, \backslash\}^2$ . It must be that  $a, b, c$  share a live block in  $T_{\text{top}}$ . If only  $b$  and  $c$  shared a live block then the projection of  $T_{\text{top}}$  onto  $\{a, b, c\}$  would need to have the form  $a^*(bc \text{ or } cb)a^*b^*c^*b^*a^*$ , violating Lemma 8.2 since neither  $(bc)$  and  $(cb)$  can be followed by  $bc$ . If only  $a$  and  $b$  shared a live block the projection onto  $\{a, b, c\}$  would need to have the form  $a^*b^*c^*(ba \text{ or } ab)c^*b^*a^*$ , which violates the property that live blocks contain only first occurrences.

We have deduced that  $a, b$ , and  $c$  share a live block  $B$  in  $T_{\text{top}}$ , but they do not necessarily appear in that order. To form a copy of  $M_2$ , some prefix must arise from substituting the type  $\pi^-$  sequence  $T_{\text{mid}}(B)$  for  $B$ ; the remaining suffix must follow  $a, b$ , and  $c$ 's live block in  $T_{\text{top}}$ . We can always include at least one symbol in the suffix, so the split between prefix and suffix can be one of three options: (i)  $abcbab|cba$ , or (ii)  $abcbabc|ba$ , or (iii)  $abcbabc|a$ . In cases (i) and (ii),  $b$  must precede  $a$  in  $B$ , meaning  $b < a$  in the canonical ordering of  $T_{\text{mid}}(B)$ . As a consequence, any occurrence of the prefix  $abcbab$  (or  $abcbabc$ ) in  $T_{\text{mid}}$  implies an occurrence of  $babbab < T_{\text{mid}}$ , contradicting Lemma 8.3. In case (iii) the prefix contains  $bcbcb$ , also contradicting Lemma 8.3.  $\square$

THEOREM 8.5. For any  $k \geq 1$ ,  $M_{2^k} \prec T_\pi$ , where  $\pi \in \prec\{/, \backslash\}^2\prec^k$ . As a consequence,  $\text{Ex}(M_{2^k}, n) = \Omega(n\alpha^{k+1}(n))$ .

*Proof.* The proof is by induction on  $k$ ; the base case is covered by Lemma 8.4. For succinctness let  $K = 2^k$ . As in the proof of Lemma 8.4 we can restrict our attention to the case where  $M_K$ , say over the alphabet  $a_1, \dots, a_{K+1}$ , arises in  $T_{\text{sub}}$  after a composition event. Moreover, we can assume  $a_1, \dots, a_{K+1}$  appear in a common live block  $B$ , so the projection of  $T_{\text{top}}$  onto  $\{a_1, \dots, a_{K+1}\}$  is  $(a_1 \cdots a_{K+1})a_1^* \cdots a_{K+1}^*$ . If substituting  $T_{\text{mid}}(B)$  for  $B$  creates an instance of  $M_K$ , some prefix must come from  $T_{\text{mid}}(B)$  and the remaining suffix from the sequence  $a_1^* \cdots a_{K+1}^*$  following  $B$ . There are two cases: either the suffix contains a strict majority of the  $K+1$  symbols or a strict minority. In the former case we have  $a_{K/2+1} < \cdots < a_{K+1}$  according to the canonical ordering of  $T_{\text{mid}}(B)$ , so any instance of the  $N$ -shaped pattern  $a_{K+1}a_K \cdots a_{K/2+1}a_{K/2+2} \cdots a_{K+1}a_K \cdots a_{K/2+1}$  in  $T_{\text{mid}}(B)$  implies that it also contains

$$M_{K/2} = a_{K/2+1} a_{K/2+2} \cdots a_{K+1} a_K \cdots a_{K/2+1} a_{K/2+2} \cdots a_{K+1} a_K \cdots a_{K/2+1},$$

which contradicts the hypothesis that  $T_{\text{mid}}$  is  $M_{K/2}$ -free. If, on the other hand, the suffix of  $M_K$  following  $B$  contains a strict minority of  $\{a_1, \dots, a_{K+1}\}$ , then  $T_{\text{mid}}(B)$  must contain an instance of  $M_{K/2}$  on the alphabet  $a_1, \dots, a_{K/2+1}$ , also contradicting the inductive hypothesis.  $\square$

We now turn to the analysis of the forbidden sequences of  $U_\pi$ .

THEOREM 8.6. For any  $k \geq 0$ ,  $Z_{3^k} \prec U_\pi$ , where  $\pi = \prec^{k+1} \swarrow \searrow^{k+1}$ . As a consequence,  $\text{Ex}(Z_{3^k}, n) > n \cdot 2^{(1+o(1))\alpha^{k+1}(n)/(k+1)!}$ .

Before proving Theorem 8.6 we must make a few observations. First, preshuffling is the norm when generating sequences of type  $\pi = \prec^{k+1} \swarrow \searrow^{k+1}$ . Whenever  $k > 0$  we have  $\pi_2 \pi_{|\pi|-1} = \prec \searrow$ , implying preshuffling is used; it is only when  $k = 0$  (pattern

$\wedge\wedge\wedge$ ) that postshuffling is used. In any sequence formed using preshuffling, if a block contains  $(a_1 \cdots a_l)$ , the projection of the sequence onto  $\{a_1, \dots, a_l\}$  is of the form

$$a_1^* a_2^* \cdots a_l^* (a_1 \cdots a_l) a_l^* a_{l-1}^* \cdots a_1^*.$$

Second, note that Theorem 8.6 fails to be true for most patterns. Indeed, sufficiently long sequences of type  $\wedge\wedge\wedge$  contain  $Z_K$  for *every*  $K > 0$ .

*Proof.* The proof is by induction on  $k$ . For succinctness we let  $K = 3^k$ . In the base case  $k = 0$ ,  $Z_K = ababab$ , and  $U_\pi = U_{\wedge\wedge\wedge}$  is *ababab*-free, by Lemma 2.4. In the general case  $k \geq 1$  and  $\pi = \wedge^{k+1} \wedge \wedge^{k+1}$ , so  $U_\pi = U_{\text{sub}} \otimes U_{\text{bot}} = (U_{\text{top}} \circ U_{\text{mid}}) \otimes U_{\text{bot}}$  is formed by composing  $U_{\text{top}}$  with  $U_{\text{mid}}$ , a type  $\pi^-$  sequence, then *preshuffling* it with  $U_{\text{bot}}$ . We can assume that any occurrence of  $Z_K$  arises from the composition event  $U_{\text{sub}} = U_{\text{top}} \circ U_{\text{mid}}$  since *ababab*  $< Z_K$  for every pair of symbols  $\{a, b\} \subset \Sigma(Z_K)$  and *ababab* cannot be introduced by shuffling. Write  $Z_K$  as

$$a_1 a_2 \cdots a_{K/3+1} a_K \cdots a_1 a_2 \cdots a_{K/3+1} a_K \cdots a_1 a_2 \cdots a_{K/3+1}.$$

It is easy to verify that if  $Z_K$  occurs in  $U_{\text{sub}}$ , it must be that  $\{a_1, \dots, a_{K/3+1}\}$  share a *single* block  $B$  in  $U_{\text{top}}$ . (Note, however, that their canonical orderings in  $U_{\text{top}}$  and  $U_{\text{mid}}(B)$  are *not* necessarily  $a_1 < \cdots < a_{K/3+1}$ .) Some prefix of  $Z_K$  appears before  $B$  in  $U_{\text{top}}$ , some suffix of  $Z_K$  after  $B$  in  $U_{\text{top}}$ , and the remaining middle portion appears in  $U_{\text{mid}}(B)$ . Suppose  $a_1 \cdots a_l$  is the prefix and  $a_{l'} a_{l'+1} \cdots a_{K/3+1}$  the suffix, for some indices  $l, l'$ . It follows that  $a_1 < a_2 < \cdots < a_l$  and  $a_{K/3+1} < a_K < \cdots < a_{l'}$  according to the canonical ordering of  $U_{\text{mid}}(B)$ , which implies  $l \leq l'$ . (Since preshuffling is used, the canonical ordering of *non-first* symbols in  $B$  is the same in  $U_{\text{top}}$  and  $U_{\text{mid}}(B)$ , though the same is not true of symbols making their first appearance in  $B$ .) At least one of the following must be true

- (i) the prefix contains at least  $K/3 + 1$  symbols and is disjoint from the suffix, that is,  $l \geq K/3 + 1$  and  $l < l'$ .
- (ii) the suffix contains at least  $K/3 + 1$  symbols and we are not in case (i), that is,  $l' \leq 2K/3 + 1$ .
- (iii) there are at least  $K/3 + 1$  symbols in neither the prefix nor suffix, that is,  $l \leq K/3$  and  $l' \geq 2K/3 + 2$ .

Case (iii) is the simplest. To form a copy of  $Z_K$  in  $U_{\text{sub}}$ , we would need  $U_{\text{mid}}(B)$  to contain a copy of  $Z_{K/3}$  on the alphabet  $\{a_{K/3+1}, \dots, a_{2K/3+1}\}$ , contradicting the inductive hypothesis. In Case (i),  $U_{\text{mid}}(B)$  must contain  $a_{K/3+1} \cdots a_1 \cdots a_{K/3+1} \cdots a_1 \cdots a_{K/3+1}$ . However, by the canonical ordering  $a_1 < \cdots < a_{K/3+1}$ , this implies that the first  $a_{K/3+1}$  is preceded by  $a_1 \cdots a_{K/3}$ , meaning  $U_{\text{mid}}(B)$  also contains a copy of  $Z_{K/3}$ , a contradiction. Case (ii) is symmetric to Case (i).  $\square$

**8.4. Comb-shaped Sequences.** The results of [9, 14, 20, 21] show that *ababa* and *abcacbc* are the only minimally non-linear 2-sparse forbidden sequences over a three-letter alphabet, both with extremal function  $\Theta(n\alpha(n))$ . Just as *ababa* can be generalized to  $M$ -shaped sequences,  $C_1 = abcacbc$  can be generalized to the one-sided *comb-shaped* sequences  $\{C_k\}_{k \geq 1}$ , where

$$C_k = 1 \ 2 \ 3 \ \cdots \quad (k+2) \quad (k+2) \quad (k+2) \quad (k+2) \quad \cdots \quad (k+1) \quad (k+2).$$

Our parameterized sequences let us obtain non-trivial lower bounds on comb-shaped sequences.

**THEOREM 8.7.** *For all  $k \geq 1$ ,  $C_k \prec T_\pi$ , where  $\pi = \swarrow \wedge^k$ . Consequently,  $\text{Ex}(C_k, n) = \Omega(n\alpha^k(n))$ .*

*Proof.* The proof is by induction on  $k$ . Theorem 8.1 (see [20]) takes care of the base case  $C_1 = abcacbc$ . We will focus on  $C_2 = abcdadbdc$ , then note why the argument works for any  $k$ . Define  $T_{\text{top}}, T_{\text{sub}}, T_{\text{bot}}, T_{\text{mid}}$ , and  $T_{\text{sh}}$  as usual, where  $T_{\text{mid}}$  is now a type  $\swarrow \wedge$  sequence. Note that both  $T_{\text{top}}$  and  $T_{\text{mid}}$  are formed using preshuffling, so if a live block in either contains  $(a_1 \cdots a_l)$ , the projection of the sequence onto  $\{a_1, \dots, a_l\}$  is of the form  $(a_1 \cdots a_l)a_l^* \cdots a_1^*$ .

We first argue that  $\{a, b, c, d\} \subset \Sigma(T_{\text{top}})$ . One may check that the only case that does not immediately violate Lemma 8.2(1) is that  $a \in \Sigma(T_{\text{bot}}^*)$  while  $b, c, d \in \Sigma(T_{\text{top}})$ . This means that for  $C_2$  to show up in  $T_{\text{sh}}$  we must already have  $(bcd)dbdc < T_{\text{sub}}$ , where the live block  $(bcd)$  is shuffled into  $a$ 's copy of  $T_{\text{bot}}$ . However, Lemma 8.2(3) implies that the projection of  $T_{\text{top}}$  onto  $\{b, c, d\}$  is  $(bcd)d^*c^*b^*$ , and therefore that the projection of  $T_{\text{sub}}$  onto  $\{b, c, d\}$  is  $(bcd)d^*c^*b^*d^*c^*b^*$ . This does not contain  $(bcd)dbdc$ . We proceed to consider the case when  $\{a, b, c, d\} \subset \Sigma(T_{\text{top}})$ .

One can see that  $a, b, c$ , and  $d$  must share a live block  $B$  in  $T_{\text{top}}$ . If the first two  $a$ s in  $C_2 < T_{\text{sub}}$  arose from the composition that created  $T_{\text{sub}}$  then  $b, c$ , and  $d$  must have been in  $a$ 's live block. If not then  $C_2$  would have already appeared in  $T_{\text{top}}$ . Thus, some prefix of  $C_2$  arose from substituting  $T_{\text{mid}}(B)$  for  $B$  and the remaining suffix followed  $B$  in  $T_{\text{top}}$ . Lemma 8.2(2) implies that the suffix cannot be  $dcd$  for otherwise  $(cd)cd < T_{\text{top}}$  or  $(dc)dc < T_{\text{top}}$ . This implies that  $abdadbdc = C_1 < T_{\text{mid}}(B)$  (a type  $\swarrow \wedge$  sequence), which contradicts Theorem 8.1.

For  $k > 2$  write  $C_k = a_1 a_2 \cdots a_{k+1} b a_1 b a_2 b \cdots b a_{k+1} b$ . The same argument from above shows that  $\{a_1, \dots, a_{k+1}, b\}$  are contained in a single block  $B$  of  $T_{\text{top}}$ . For  $C_k$  to arise in  $T_{\text{sub}}$  a prefix of it must come from  $T_{\text{mid}}(B)$  and a suffix from the part of  $T_{\text{top}}$  following  $B$ . By Lemma 8.2(2) the suffix cannot be  $b a_{k+1} b$ , which means the prefix in  $T_{\text{mid}}(B)$  must contain  $a_1 \cdots a_k b a_1 b a_2 b \cdots b a_k b = C_{k-1}$ , contradicting the inductive hypothesis.  $\square$

**9. Conclusions.** In Theorem 1.3 we established sharp bounds on the functions  $\Lambda_{r,s}$  and  $\Lambda_{r,s}^{\text{dbl}}$ , for all values of  $r$  and  $s$ , and showed, perhaps surprisingly, that these extremal functions are essentially the same. Moreover, they match  $\lambda_s$  and  $\lambda_s^{\text{dbl}}$  only when  $s \leq 3$ , or  $s \geq 4$  is even, or  $r = 2$ . However, Theorem 1.3 is *not* the last word on  $\Lambda_{r,s}^{\text{dbl}}$ . In Cibulka and Kynčl's [3] application of  $\Lambda_{r,s}^{\text{dbl}}(n, m)$ ,  $s$  is a fixed parameter whereas  $r$  is variable and cannot be bounded as a function of  $s$ . Cibulka and Kynčl require upper bounds on  $\Lambda_{r,s}^{\text{dbl}}(n, m)$  that are linear in  $r$  whereas the leading constant in our bounds matches that of  $\Lambda_{r,2}^{\text{dbl}}(n, m)$ , currently known to be at most  $O(6^r)$ . See Lemma 3.3. In other words, we now have two incomparable upper bounds on  $\Lambda_{r,2}^{\text{dbl}}(n, m)$  when  $r$  is not treated as a constant, namely  $O((n+rm)\alpha(n, m))$  [3], which has optimal dependence on  $r$ , and  $O(6^r(n+m))$ , which is optimal for fixed  $r$ . Whether  $\Lambda_{r,2}^{\text{dbl}}(n, m) = O(n+rm)$  or not is an intriguing open question.

We have shown that *doubling* various forbidden patterns (alternating sequences and  $(r, s+1)$ -formations) has no significant effect on their extremal functions. It is an open problem whether  $\text{Ex}(\text{dbl}(\sigma), n)$  is asymptotically equivalent to  $\text{Ex}(\sigma, n)$  for every  $\sigma$ . We conjecture the answer is no when  $\sigma$  can be a *set* of forbidden sequences, though it seems plausible the answer is yes for any single forbidden sequence.

**CONJECTURE 9.1.** *In general, it is not true that  $\text{Ex}(\text{dbl}(\sigma), n) = \Theta(\text{Ex}(\sigma, n))$ . In particular, whereas  $\text{Ex}(\text{dbl}(\{ababa, abcacbc\}), n) = \Theta(n\alpha(n))$ , we conjecture that*

$\text{Ex}(\{ababa, abcacbc\}, n) = O(n)$ .

The main open problem in the realm of generalized Davenport–Schinzel sequences is to characterize linear forbidden sequences, or equivalently, to enumerate all minimally non-linear forbidden sequences. The number of minimally non-linear sequences (with respect to the partial order  $<$ ) is almost certainly infinite [20], but whether there are infinitely many *genuinely* different non-linear sequences is open. Refer to [20] for a discussion of how “genuinely” might be formally defined.

CONJECTURE 9.2. (*Informal*) *Every nonlinear sequence  $\sigma$  (having  $\text{Ex}(\sigma, n) = \omega(n)$ ) contains  $ababa, abcacbc$ , or some sequence morally equivalent to  $abcacbc$ .*

Our lower bounds on  $\text{Ex}(M_k, n)$  are weak, as a function of  $k$ , and we have provided no non-trivial upper bounds. It may be possible to generalize the proof of Theorem 6.9 to show  $\text{Ex}(M_k, n) = O(n \text{ poly}(\alpha(n)))$ , where the degree of the polynomial depends on  $k$ .

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## Appendix A. Proofs.

**A.1. Proof of Lemma 1.2.** Recall that  $\text{dbl}(\text{Form}(r, s + 1)) = \{\text{dbl}(\sigma) \mid \sigma \in \text{Form}(r, s + 1)\}$  whereas sequences in  $\text{dblForm}(r, s + 1)$  are formed by taking the concatenation of  $s + 1$  sequences, the first and last being a permutation of  $\{1, \dots, r\}$  and all the rest containing two occurrences of  $\{1, \dots, r\}$ . For example,  $abc\ abacbbca \in \text{dblForm}(3, 3)$  whereas  $abbcc\ cbbba\ bbcca \in \text{dbl}(\text{Form}(3, 3))$ . We restate Lemma 1.2.

LEMMA 1.2. *The following bounds hold for any  $r \geq 2, s \geq 1$ .*

$$\begin{aligned} \text{Ex}(\text{dbl}(\text{Form}(r, s + 1)), n, m) &\leq r \cdot \Lambda_{r,s}^{\text{dbl}}(n, m) + 2rn \\ \text{Ex}(\text{dbl}(\text{Form}(r, s + 1)), n) &= O(\Lambda_{r,s}^{\text{dbl}}(n)). \end{aligned}$$

*Proof.* Let  $S$  be a  $\text{dbl}(\text{Form}(r, s + 1))$ -free sequence over an  $n$ -letter alphabet. Obtain  $S'$  from  $S$  by discarding the first occurrence and last  $r$  occurrences of each letter, then retaining every  $r$ th occurrence of each letter (i.e., the  $r$ th,  $2r$ th,  $3r$ th, etc.), discarding the rest. Clearly  $S'$  has the property that each  $b$  is preceded and followed by at least  $r$   $b$ s in  $S$ , and between two  $b$ s in  $S'$  there are at least  $r - 1$   $b$ s in  $S$ . It follows that  $|S'| \geq (|S| - 2rn)/r$ . Suppose  $|S'|$  contained some sequence  $\sigma'_1 \cdots \sigma'_{s+1} \in \text{dblForm}(r, s + 1)$ . (Recall that  $\sigma'_1$  and  $\sigma'_{s+1}$  contain one copy of  $\{1, \dots, r\}$  whereas  $\sigma'_2, \dots, \sigma'_s$  contain two copies of  $\{1, \dots, r\}$ .) This implies that  $S$  contains a sequence  $\sigma_1 \cdots \sigma_{s+1}$  where each  $\sigma_k$  contains  $r + 1$  copies of  $\{1, \dots, r\}$ . We claim each  $\sigma_k$  contains a doubled permutation of  $\{1, \dots, r\}$ , which implies that  $S$  is not  $\text{dbl}(\text{Form}(r, s + 1))$ -free, a contradiction. Find the symbol  $b$  in  $\sigma_k$  whose second occurrence is earliest, that is, we can write  $\sigma_k = \sigma'_k b \sigma''_k b \sigma'''_k$ , where  $\sigma'_k \sigma''_k$  contains at most one copy of each symbol. Since  $\sigma'''_k$  contains at least  $r$  copies of the  $r - 1$  symbols in  $\{1, \dots, r\} \setminus \{b\}$  we can continue to find a doubled permutation of  $\{1, \dots, r\} \setminus \{b\}$  by induction. If  $S$  is an  $m$ -block sequence then  $S'$  is too, giving the first bound. When  $S$  is merely  $r$ -sparse we can only bound  $S'$  by  $\Lambda_{r,s}^{\text{dbl}}(n)$  if it, too, is  $r$ -sparse. This is done as follows.

Greedily partition  $S = S_1 S_2 \dots S_m$  into maximal sequences  $\{S_q\}$  over alphabets of size exactly  $2r^2$ , with  $\|S_m\|$  perhaps smaller. Since each  $S_q$  has length at most  $\text{Ex}(\text{dbl}(\text{Form}(r, s+1)), 2r^2) = O(1)$ , it follows that  $m = \Omega(|S|)$ . Obtain  $T$  by replacing each  $S_q$  with a block consisting of its alphabet  $\Sigma(S_q)$ . If  $|T| \leq 2r^2 n$  there is nothing to prove since  $|S| = \Theta(|T|) = O(n) = O(\Lambda_{r,s}^{\text{dbl}}(n))$ , so assume otherwise. Obtain  $T'$  from  $T$  by discarding the first occurrence and last  $r$  occurrences of each letter, then retaining every  $r$ th occurrence of each letter. It follows that  $|T'| \geq (|T| - 2rn)/r \geq |T| \frac{r-1}{r^2}$ , that is, the average length of blocks in  $T'$  is at least  $2(r-1)$ . Let  $T''$  be an  $r$ -sparse subsequence of  $T'$  obtained by scanning  $T'$  from left to right, removing a symbol if it is identical to one of the preceding  $r-1$  symbols. At most  $r-1$  letters from each block of  $T'$  can be removed in this process. The average block length of  $T''$  is at least  $2(r-1) - (r-1) \geq 1$ , hence  $|T''| \geq m = \Omega(|S|)$ . Since  $T''$  is  $\text{dblForm}(r, s+1)$ -free, we have  $|S| = O(\Lambda_{r,s}^{\text{dbl}}(n))$ .  $\square$

**A.2. Proof of Lemma 3.1.** There is no theorem to the effect that  $\text{Ex}(\sigma, n) = O(\text{Ex}(\sigma, n, O(n)))$ . Lemma 3.1 restates the best known reductions from  $r$ -sparse to blocked sequences. Some ad hoc reductions are known to be superior, for example, those for order-5 DS sequences [17, §6.2].

LEMMA 3.1. (*Cf. Sharir [24], Füredi and Hajnal [7], and Pettie [17].*) Define  $\gamma_s, \gamma_s^{\text{dbl}}, \gamma_{r,s}, \gamma_{r,s}^{\text{dbl}} : \mathbb{N} \rightarrow \mathbb{N}$  to be non-decreasing functions bounding the leading factors of  $\lambda_s(n), \lambda_s^{\text{dbl}}(n), \Lambda_{r,s}(n)$ , and  $\Lambda_{r,s}^{\text{dbl}}(n)$ , e.g.,  $\Lambda_{r,s}^{\text{dbl}} \leq \gamma_{r,s}^{\text{dbl}}(n) \cdot n$ . The following bounds hold.

$$\begin{aligned} \lambda_s(n) &\leq \gamma_{s-2}(n) \cdot \lambda_s(n, 2n) \\ \lambda_s^{\text{dbl}}(n) &\leq (\gamma_{s-2}^{\text{dbl}}(n) + 4) \cdot \lambda_s^{\text{dbl}}(n, 2n) \\ \lambda_s(n) &\leq \gamma_{s-2}(\gamma_s(n)) \cdot \lambda_s(n, 3n) \\ \lambda_s^{\text{dbl}}(n) &\leq (\gamma_{s-2}^{\text{dbl}}(\gamma_s^{\text{dbl}}(n)) + 4) \cdot \lambda_s^{\text{dbl}}(n, 3n) \\ \Lambda_{r,s}(n) &\leq \gamma_{r,s-2}(n) \cdot \Lambda_{r,s}(n, 2n) + 2n \\ \Lambda_{r,s}^{\text{dbl}}(n) &\leq (\gamma_{r,s-2}^{\text{dbl}}(n) + O(1)) \cdot \Lambda_{r,s}^{\text{dbl}}(n, 2n) \\ \Lambda_{r,s}(n) &\leq \gamma_{r,s-2}(\gamma_{r,s}(n)) \cdot \Lambda_{r,s}(n, 3n) + 2n \\ \Lambda_{r,s}^{\text{dbl}}(n) &\leq (\gamma_{r,s-2}^{\text{dbl}}(\gamma_{r,s}^{\text{dbl}}(n)) + O(1)) \cdot \Lambda_{r,s}^{\text{dbl}}(n, 3n), \end{aligned}$$

where the  $O(1)$  terms depend on  $r$  and  $s$ .

*Proof.* All the bounds are obtained from the following sequence manipulations, which were first used by Hart and Sharir [9] and Sharir [24]. Let  $S$  be an  $r$ -sparse sequence avoiding some set  $\sigma$  of subsequences over an  $r$ -letter alphabet, so  $|S| \leq \text{Ex}(\sigma, n)$ . Greedily parse  $S$  into  $m$  intervals  $S_1 S_2 \dots S_m$  by choosing  $S_1$  to be the maximum-length prefix satisfying some property  $\mathcal{P}$ ,  $S_2$  to be the maximum-length prefix of the remaining sequence satisfying  $\mathcal{P}$ , and so on. Form  $S' = \Sigma(S_1) \Sigma(S_2) \dots \Sigma(S_m)$  by replacing each interval  $S_i$  with a single block  $\Sigma(S_i)$  containing its alphabet, listed in order of first appearance. Since  $S'$  is a subsequence of  $S$ ,  $|S'| \leq \text{Ex}(\sigma, n, m)$ . To bound  $|S|$  we only need to determine upper bounds on  $m$  and the *shrinkage* factor  $|S|/|S'|$ .

*Bounds on  $\lambda_s$ .* If we parse  $S$  into maximal order- $(s-2)$  sequences then each  $S_i$  must contain either the first or last occurrence of some symbol, hence  $m \leq 2n$ . The shrinkage factor is  $|S_i|/\|S_i\| \leq \gamma_{s-2}(\|S_i\|) \leq \gamma_{s-2}(n)$ , which gives the first inequality. Now consider parsing  $S$  into  $m$  maximal sequences that are both order- $(s-2)$  DS

sequences *and* have length at most  $\gamma_s(n)$ . It follows that  $m \leq 3n$ : at most  $n$  sequences were terminated because they reached length  $\gamma_s(n)$  (by definition of  $\gamma_s$ ) and the remaining sequences number at most  $2n$  since each must contain the first or last occurrence of some letter.

*Bounds on  $\lambda_s^{\text{dbl}}$ .* Let  $\sigma_{s+2}$  be the alternating sequence with length  $s+2$ . Order- $s$  double DS sequences are  $\text{dbl}(\sigma_{s+2})$ -free. Obtain  $\sigma'_{s+2}$  by doubling each letter of  $\sigma_{s+2}$ , including the first and last. It is easy to show that  $\text{Ex}(\sigma'_{s+2}, n) \leq \lambda_s^{\text{dbl}}(n) + 4n$  so we can take  $\gamma_s^{\text{dbl}}(n) + 4$  to be the leading factor in this extremal function. Consider parsing an order- $s$  double DS sequence  $S$ . If we parse  $S$  into maximal  $\sigma'_s$ -free sequences then each subsequence must contain the first or last occurrence of some symbol, so  $m \leq 2n$  and the shrinkage factor is at most  $\gamma_{s-2}^{\text{dbl}}(n) + 4$ . If, further, we truncate any subsequence in the parsing at length  $\gamma_s^{\text{dbl}}(n)$ , then  $m \leq 3n$  and the shrinkage factor is at most  $\gamma_{s-2}^{\text{dbl}}(\gamma_s^{\text{dbl}}(n)) + 4$ .

*Bounds on  $\Lambda_{r,s}$  and  $\Lambda_{r,s}^{\text{dbl}}$ .* The argument is the same, except that during the parsing step, we discard any symbol that triggers the termination of a subsequence. For example, if  $S$  is a  $\text{Form}(r, s+1)$ -free sequence we parse it into  $S_1 a_1 S_2 a_2 \cdots a_{m-1} S_m a_m$ , where the  $\{S_i\}$  are maximal  $\text{Form}(r, s-1)$ -free sequences and  $\{a_i\}$  the single letters following them, where  $a_m$  might not be present. Since  $S_i a_i$  contains some element of  $\text{Form}(r, s-1)$ ,  $S_i a_i$  must contain the first or last occurrence of some letter, hence  $m \leq 2n$ . We form  $S'$  by contracting each  $S_i$  to a single block, discarding  $a_i$ , so the shrinkage factor is at most  $\gamma_{r,s-2}(n)$ . It follows that  $|S| \leq \gamma_{r,s-2}(n) \cdot \Lambda_{r,s}(n, 2n) + 2n$ . The procedure for  $\Lambda_{r,s}^{\text{dbl}}$  is a straightforward combination of the procedures described above, for  $\Lambda_{r,s}$  and  $\lambda_s^{\text{dbl}}$ .  $\square$

**A.3. Proof of Lemma 3.2.** We restate the lemma.

LEMMA 3.2. *The following inequalities hold for all  $s$ .*

$$\begin{aligned} \lambda_s(n) &\leq \Lambda_{2,s}(n) &\leq \lambda_s^{\text{dbl}}(n) &\leq \Lambda_{2,s}^{\text{dbl}}(n) + 2n &\leq 5 \cdot \lambda_s^{\text{dbl}}(n) + 2n \\ \lambda_s(n, m) &\leq \Lambda_{2,s}(n, m) &\leq \lambda_s^{\text{dbl}}(n, m) &\leq \Lambda_{2,s}^{\text{dbl}}(n, m) + n &\leq 3 \cdot \lambda_s^{\text{dbl}}(n, m) + n. \end{aligned}$$

*Proof.* Order- $s$  DS sequences are  $\text{Form}(2, s+1)$ -free, which gives the 1st and 5th inequalities.  $\text{Form}(2, s+1)$ -free sequences, in turn, are order- $s$  *double* DS sequences, which gives the 2nd and 6th inequalities. Let  $S$  be an order- $s$  double DS sequence. Form  $S' < S$  by (i) removing the first occurrence of each letter, and, if necessary, (ii) removing up to  $n$  additional symbols to restore 2-sparseness. Clearly  $|S| \leq |S'| + n$  if only (i) is applied and  $|S| \leq |S'| + 2n$  if (i) and (ii) are applied. Suppose  $S'$  contained a  $\text{dblForm}(2, s+1)$  pattern of the form  $\{ab\} \{aabb\}^{s-1} \{ab\}$ , where the bracketed sequences can be permuted arbitrarily. Together with the initial  $a$  and  $b$  in  $S$ , this shows that  $S$  contains a doubled alternating sequence isomorphic to  $abbaabb \cdots$  ( $s+2$  alternations), a contradiction. This gives the 3rd and 7th inequalities.

We now turn to the 4th and 8th inequalities. Let  $S$  be a 2-sparse  $\text{dblForm}(2, s+1)$ -free sequence and let  $S'$  be derived as follows.

- (i) Retain every third occurrence of each letter, starting from the first; discard all others.
- (ii) Discard additional occurrences to restore 2-sparseness.

The number of letters discarded in step (i) is clearly at most  $(2/3)|S|$ . We claim the number discarded in step (ii) is at most  $1/5$ th the number discarded in step (i). Suppose we see two consecutive  $as$  after step (i), one of which will be removed to restore 2-sparseness. There must have been two additional  $as$  between those  $as$

removed by step (i), and by 2-sparseness, at least three non- $a$  interstitial letters, also removed by step (i). The picture looks like  $\overline{a}xaya\overline{z}\overline{a}$ , where the overlined  $a$ s are those remaining after step (i). (Obviously  $x, y$ , and  $z$  cannot all be identical, for otherwise at least one would be retained in step (i).) Thus, the total number of letters removed by steps (i) and (ii) is at most  $(6/5)(2/3)|S| = (4/5)|S|$ , so  $|S| \leq 5|S'|$ . Suppose that  $S'$  contained a doubled alternating sequence  $abbaabb \cdots$  with  $s + 2$  alternations. This implies that  $S$  contains  $\underline{abbb}\underline{aaaa}\underline{bbbb} \cdots$ , where the underlined letters appear in  $S$  but not  $S'$ . This contradicts the  $\text{dblForm}(2, s + 1)$ -freeness of  $S$ . The 4th inequality follows. The 8th follows from the same argument, omitting step (ii) in the construction of  $S'$ .  $\square$

**A.4. Proof of Lemma 3.3.** Some of the results cited in Lemma 3.3 refer to (or implicitly use) results on forbidden 0-1 matrices. See Füredi and Hajnal [7] and Pettie [19, 20, 21] for more details on the connection between matrices and sequences.

LEMMA 3.3. *At orders  $s = 1$  and  $s = 2$ , the extremal functions  $\lambda_s, \lambda_s^{\text{dbl}}, \Lambda_{r,s}$ , and  $\Lambda_{r,s}^{\text{dbl}}$  obey the following.*

$$\begin{array}{lll}
 \lambda_1(n) = n & \lambda_1(n, m) = n + m - 1 & \\
 \lambda_2(n) = 2n - 1 & \lambda_2(n, m) = 2n + m - 2 & [4] \\
 \lambda_1^{\text{dbl}}(n) = 3n - 2 & \lambda_1^{\text{dbl}}(n, m) = 2n + m - 2 & [5, 13] \\
 \lambda_2^{\text{dbl}}(n) < 8n & \lambda_2^{\text{dbl}}(n, m) < 5n + m & [11, 7] \\
 \Lambda_{r,1}(n) = \Lambda_{r,1}^{\text{dbl}}(n) < rn & \Lambda_{r,1}(n, m) = \Lambda_{r,1}^{\text{dbl}}(n, m) < n + (r - 1)m & [10] \\
 \Lambda_{r,2}(n) < 2rn & \Lambda_{r,2}(n, m) < 2n + (r - 1)m & [10] \\
 \Lambda_{r,2}^{\text{dbl}}(n) < 6^r rn & \Lambda_{r,2}^{\text{dbl}}(n, m) < 2 \cdot 6^{r-1}(n + m/3) & [21]
 \end{array}$$

*Proof.* Davenport and Schinzel [4] noted the bounds on  $\lambda_1(n)$  and  $\lambda_2(n)$ ; their extension to blocked sequences is trivial. In an overlooked note Davenport and Schinzel [4] observed without proof that  $\lambda_1^{\text{dbl}}(n) = 3n - 2$ , which was formally proved by Klazar [13]. Its extension to blocked sequences is also trivial. Adamec, Klazar, and Valtr [1] proved that  $\lambda_2^{\text{dbl}}(n) = O(n)$  and Klazar [11] bounded the leading constant between 7 and 8. A blocked sequence  $S$  can be represented as a 0-1 incidence matrix  $A_S$  whose rows correspond to symbols and columns to blocks, where  $A_S(i, j) = 1$  if and only if symbol  $i$  appears in block  $j$ . A forbidden sequence becomes a forbidden 0-1 pattern. The bound on  $\lambda_2^{\text{dbl}}(n, m)$  follows from Füredi and Hajnal's [7] analysis of a certain 0-1 pattern. The bounds on  $\Lambda_{r,1}$  and  $\Lambda_{r,2}$  were noted by Klazar [10] and Nivasch [16]. They are straightforward to prove.

Since the  $N$ -shaped sequence  $12 \cdots rr(r-1) \cdots 112 \cdots r$  over  $r$  letters is contained in  $\text{Form}(r, 3)$ , the linear upper bound on  $\text{Ex}(\text{dbl}(12 \cdots rr(r-1) \cdots 112 \cdots r), n)$  due to Klazar and Valtr [14] (see also [21]) immediately extend to  $\Lambda_{r,2}^{\text{dbl}}(n)$ . With some care the leading constants of  $\Lambda_{r,2}^{\text{dbl}}(n)$  and  $\Lambda_{r,2}^{\text{dbl}}(n, m)$  can be made reasonably small using the 0-1 matrix representation of (forbidden) sequences from [21]. Consider an  $m$ -block,  $\text{dblForm}(r, 3)$ -free sequence  $S$ . Without loss of generality assume the alphabet  $\Sigma(S) = \{1, \dots, n\}$  is ordered according to their first appearance in  $S$ . Let  $A_S$  be an  $n \times m$  0-1 matrix where  $A_S(i, j) = 1$  if and only if symbol  $i$  appears in block  $j$ . By virtue of being  $\text{dblForm}(r, 3)$ -free,  $A_S$  does not contain  $P$  as a submatrix,<sup>17</sup> where  $P$

<sup>17</sup>In this context a submatrix is obtained by deleting rows and columns from  $A_S$ , and possibly flipping some 1s to 0s.

