Overview

- **Lecture 1:**
  - Introduction to linear programming and the simplex algorithm.
  - Pivoting rules.
  - The **RandomFacet** pivoting rule.

- **Lecture 2:**
  - The Hirsch conjecture.
  - Introduction to Markov decision processes (MDPs).
  - Upper bound for the **LargestCoefficient** pivoting rule for MDPs.

- **Lecture 3:**
  - Lower bounds for pivoting rules utilizing MDPs. Example: **Bland’s rule**.
  - Lower bound for the **RandomEdge** pivoting rule.
  - Abstractions and related problems.
maximize $\mathbf{c}^T \mathbf{x}$
subject to $A\mathbf{x} \leq \mathbf{b}$
A **convex polytope** (or polyhedron) $P$ in dimension $d$ is a set of points

$$P = \{ x \in \mathbb{R}^d \mid Ax \leq b \}$$

where $A$ is an $n \times d$ matrix and $b$ is a vector in $\mathbb{R}^n$. 

$P$ is bounded if there exists a constant $K$ such that for all $x \in P$, the absolute value of every component $x_i$ is at most $K$. Otherwise $P$ is unbounded.
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I.e., $P$ is the intersection of $n$ halfspaces $a_i^T x \leq b_i$, where $a_i^T$ is the $i$'th row of $A$. 

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$P$ is **bounded** if there exists a constant $K$ such that for all $x \in P$, the absolute value of every component $x_i$ is at most $K$. Otherwise $P$ is **unbounded**.
A point $x \in \mathbb{R}^d$ is a **basic solution** if it satisfies $d$ linearly independent constraints, $a_i^T x \leq b_i$, with equality.
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\( x \in P \) is a **vertex** (or corner) if there exists a vector \( c \in \mathbb{R}^d \) such that for all \( y \in P \), if \( y \neq x \) then \( c^T x > c^T y \).
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$x \in P$ is a **vertex** (or corner) if there exists a vector $c \in \mathbb{R}^d$ such that for all $y \in P$, if $y \neq x$ then $c^T x > c^T y$.

Every basic feasible solution $x$ is a vertex of $P$. 
If $P$ is bounded, $P$ can be equivalently defined as the convex hull of its vertices.
Convex polytopes

- If $P$ is bounded, $P$ can be equivalently defined as the convex hull of its vertices.
- A $k$-face is a $k$ dimensional polytope defined by a set of vertices that satisfy the same $d - k$ constraints with equality.
  - A 0-face is a vertex.
  - A 1-face is an edge.
  - A $(d - 1)$-face is a facet.
Convex polytopes

- If $P$ is bounded, $P$ can be equivalently defined as the convex hull of its vertices.
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  - A 0-face is a vertex.
  - A 1-face is an edge.
  - A $(d - 1)$-face is a facet.
- Alternatively, a $k$-face is the polytope obtained by eliminating $d - k$ variables using the $d - k$ constraints that are satisfied with equality.
A **linear program** (LP) is the optimization problem:

\[
\begin{align*}
\text{maximize} & \quad c^T x \\
\text{s.t.} & \quad Ax \leq b \\
\end{align*}
\]
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$$

For simplicity, we generally assume that a linear program is in **canonical form**:

$$
\text{maximize } c^T x \\
\text{s.t. } Ax \leq b \\
x \geq 0
$$

Every linear program has an equivalent canonical form.
A constraint $a_i^T x \leq b_i$ can be expressed equivalently as $(a_i^T x) + s_i = b_i$, where $s_i \geq 0$ is a non-negative slack variable.
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A canonical form linear program can be transformed to **equational form** (or **standard form**) by introducing $n$ slack variables:

$$
\begin{align*}
\text{maximize} & \quad c^T x \\
\text{s.t.} & \quad A x \leq b \\
& \quad x \geq 0
\end{align*}
$$

$$
\begin{align*}
\text{maximize} & \quad c^T x \\
\text{s.t.} & \quad A x + l s = b \\
& \quad x, s \geq 0
\end{align*}
$$

The resulting linear program has $m = d + n$ non-negative variables.
A constraint \( a_i^T x \leq b_i \) can be expressed equivalently as 
\((a_i^T x) + s_i = b_i\), where \( s_i \geq 0 \) is a non-negative slack variable.

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The resulting linear program has \( m = d + n \) non-negative variables.

Note that an inequality from the original linear program is satisfied with equality if the corresponding (slack) variable is zero in the equational form.
Consider a linear program in equational form, defined by an $n \times m$ matrix $A$, with $m = d + n$:

$$\text{maximize } c^T x$$

$$s.t. \quad Ax = b$$

$$x \geq 0$$
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$$\begin{align*}
\text{maximize} & \quad c^T x \\
\text{s.t.} & \quad Ax = b \\
& \quad x \geq 0
\end{align*}$$

A **basis** is a subset $B \subseteq \{1, \ldots, m\}$ of $n$ linearly independent columns of $A$. 
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Every basis \( B \) defines a basic solution \( x_B \in \mathbb{R}^m \), as the unique solution to:

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A x = b \quad \text{and} \quad \forall i \notin B : x_i = 0
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Every basis $B$ defines a basic solution $x_B \in \mathbb{R}^m$, as the unique solution to:

$$Ax = b \quad \text{and} \quad \forall i \notin B : x_i = 0$$

Every basic solution $x \in \mathbb{R}^m$ is defined by at least one basis. If $x$ is defined by more than one basis, $x$ is a **degenerate** basic solution.
A basic solution $x_B$ is feasible if $x_B \geq 0$. 
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For some basis \( B \), the variables \( x_i \), for \( i \in B \), are called \textbf{basic}, and the remaining variables are called \textbf{non-basic}. 
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If $x_B$ is a basic feasible solution, then the non-basic variables correspond to facets defining the vertex.
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The operation of exchanging a single basic variable in $B$ with a non-basic variable, producing a new basis $B'$, is called pivoting.
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- Geometrically, if $x_B$ and $x_{B'}$ are different basic feasible solutions, pivoting corresponds to moving from $x_B$ to $x_{B'}$ along an edge of the polytope.
The simplex algorithm, Dantzig (1947):

- Start with some basis $B$ corresponding to a basic feasible solution $x_B$.
- Repeatedly perform pivots leading to new bases $B'$ corresponding to basic feasible solutions $x_{B'}$ with better values, $c^T x_{B'} \geq c^T x_B$.
- Stop when no pivot can increase the value further.
Example: The tableau method

max \quad 2x_1 - 2x_2 - x_3 \\
\text{s.t.} \quad \frac{1}{3}x_1 - \frac{2}{3}x_2 - \frac{2}{3}x_3 \leq 1 \\
\quad x_2 + x_3 \leq 2 \\
\quad x_3 \leq 1 \\
\quad x_1, x_2, x_3 \geq 0

- Transform a linear program in canonical form to equational form by introducing slack variables.
Example: The tableau method

\begin{align*}
\text{max} & \quad 2x_1 - 2x_2 - x_3 \\
\text{s.t.} & \quad \frac{1}{3}x_1 - \frac{2}{3}x_2 - \frac{2}{3}x_3 + x_4 = 1 \\
& \quad x_2 + x_3 + x_5 = 2 \\
& \quad x_3 + x_6 = 1 \\
& \quad x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
\end{align*}

- Transform a linear program in canonical form to equational form by introducing slack variables.
Example: The tableau method

\[
\begin{align*}
\text{max} & \quad 2x_1 - 2x_2 - x_3 \\
\text{s.t.} & \quad x_4 = 1 - \frac{1}{3}x_1 + \frac{2}{3}x_2 + \frac{2}{3}x_3 \\
& \quad x_5 = 2 - x_2 - x_3 \\
& \quad x_6 = 1 - x_3 \\
& \quad x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
\end{align*}
\]

- Transform a linear program in canonical form to equational form by introducing slack variables.
- Pick a basis, in this case \{4, 5, 6\}, and express the basic variables and the objective function in terms of non-basic variables. This representation is called a \textbf{tableau}. 
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\begin{align*}
\text{max} & \quad 2x_1 - 2x_2 - x_3 \\
\text{s.t.} & \quad x_4 = 1 - \frac{1}{3}x_1 + \frac{2}{3}x_2 + \frac{2}{3}x_3 \\
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& \quad x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
\end{align*}
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- Transform a linear program in canonical form to equational form by introducing slack variables.
- Pick a basis, in this case \(\{4, 5, 6\}\), and express the basic variables and the objective function in terms of non-basic variables. This representation is called a \textit{tableau}.
- The corresponding basic solution and its value can be read by setting the non-basic variables to zero.

\[
x^T = (0, 0, 0, 1, 2, 1)
\]
Example: The tableau method

\[
\begin{align*}
\text{max} & \quad 2x_1 - 2x_2 - x_3 \\
\text{s.t.} & \quad x_4 = 1 - \frac{1}{3}x_1 + \frac{2}{3}x_2 + \frac{2}{3}x_3 \\
& \quad x_5 = 2 - x_2 - x_3 \\
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& \quad x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
\end{align*}
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- If the coefficient of a non-basic variable \( x_i \) in the objective function is positive, increasing \( x_i \) will improve the value.

\[ x^T = (0, 0, 0, 1, 2, 1) \]
Example: The tableau method

\[
\begin{align*}
\text{max} & \quad 2x_1 - 2x_2 - x_3 \\
\text{s.t.} & \quad x_4 = 1 - \frac{1}{3}x_1 + \frac{2}{3}x_2 + \frac{2}{3}x_3 \\
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& \quad x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
\end{align*}
\]

- If the coefficient of a non-basic variable \(x_i\) in the objective function is positive, increasing \(x_i\) will improve the value.
- \(x_i\) can be increased until another basic variable \(x_j\) becomes zero, which completes the pivot. The basis is then updated by exchanging \(i\) and \(j\). If no variable becomes zero the value is unbounded.
Example: The tableau method

\[
\begin{align*}
\text{max} & \quad 6 + 2x_2 + 3x_3 - 6x_4 \\
\text{s.t.} & \quad x_1 = 3 + 2x_2 + 2x_3 - 3x_4 \\
& \quad x_5 = 2 - x_2 - x_3 \\
& \quad x_6 = 1 - x_3 \\
& \quad x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
\end{align*}
\]

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Example: The tableau method

\[
\begin{align*}
\text{max} & \quad 10 + x_3 - 6x_4 - 2x_5 \\
\text{s.t.} & \quad x_1 = 7 - 3x_4 - 2x_5 \\
& \quad x_2 = 2 - x_3 - x_5 \\
& \quad x_6 = 1 - x_3 \\
& \quad x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
\end{align*}
\]

- If the coefficient of a non-basic variable \( x_i \) in the objective function is positive, increasing \( x_i \) will improve the value.
- \( x_i \) can be increased until another basic variable \( x_j \) becomes zero, which completes the pivot. The basis is then updated by exchanging \( i \) and \( j \). If no variable becomes zero the value is **unbounded**.
Example: The tableau method

\[
\begin{align*}
\text{max} \quad & 11 - 6x_4 - 2x_5 - x_6 \\
\text{s.t.} \quad & x_1 = 7 - 3x_4 - 2x_5 \\
& x_2 = 1 - x_5 + x_6 \\
& x_3 = 1 - x_6 \\
& x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
\end{align*}
\]

- If the coefficient of a non-basic variable \( x_i \) in the objective function is positive, increasing \( x_i \) will improve the value.
- \( x_i \) can be increased until another basic variable \( x_j \) becomes zero, which completes the pivot. The basis is then updated by exchanging \( i \) and \( j \). If no variable becomes zero the value is **unbounded**.
- When all coefficients are negative, the solution is optimal.
The tableau method, formally

- Let $B$ be a basis, and let $A = [A_B \mid A_{\bar{B}}]$ and $x^T = [x_B^T \mid x_{\bar{B}}^T]$. I.e., $A_B$ is the matrix of basic columns and $A_{\bar{B}}$ is the matrix of non-basic columns, and similar for $x$.

- The tableau method rewrites the linear program:

\[
\begin{align*}
\max & \quad c^T x \\
\text{s.t.} & \quad Ax = b \\
\quad & \quad x \geq 0
\end{align*}
\]

\[
\begin{align*}
\max & \quad c_B^T A_B^{-1} b + \bar{c}^T x \\
\text{s.t.} & \quad x_B = A_B^{-1} b - A_B^{-1} A_{\bar{B}} x_{\bar{B}} \\
\quad & \quad x \geq 0
\end{align*}
\]

where $\bar{c} \in \mathbb{R}^m$ is the vector of reduced costs:

$$
\bar{c} = c - (A_B^{-1} A)^T c_B
$$
Two choices must be made when pivoting:

1. Which non-basic variable with positive coefficient enters the basis?
2. Which basic variable leaves the basis, in case of a tie?

These choices are specified by a pivoting rule.
Pivoting rules

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- **LargestCoefficient**, Dantzig (1947)
  - The non-basic variable with largest coefficient enters the basis.
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- **LargestCoefficient, Dantzig (1947)**
  - The non-basic variable with largest coefficient enters the basis.

- **LargestIncrease**
  - The non-basic variable that gives the largest increase enters the basis.
Pivoting rules

Two choices must be made when pivoting:

1. Which non-basic variable with positive coefficient enters the basis?
2. Which basic variable leaves the basis, in case of a tie?

These choices are specified by a pivoting rule.

- **LARGEST COEFFICIENT, Dantzig (1947)**
  - The non-basic variable with largest coefficient enters the basis.

- **LARGEST INCREASE**
  - The non-basic variable that gives the largest increase enters the basis.

- **STEEPEST EDGE**
  - The non-basic variable whose pivot corresponds to the edge with direction closest to \( c \) enters the basis.
Pivoting rules

- If the current basic feasible solution is degenerate, it is possible that the value does not increase when pivoting.
- Such situations may lead to cycling. The following two pivoting rules do not cycle, however.

- **Bland’s rule**, Bland (1977)
  - Always pick the available variable with the smallest index, both for entering and leaving the basis.

- **Lexicographic rule**, Dantzig, Orden and Wolfe (1955)
  - Pick any variable $x_i$ with positive coefficient for entering the basis.
  - Pick $x_j$ for leaving the basis such that the right-hand-side coefficients in the tableau are lexicographically smallest when divided by the coefficient of $x_i$ in that row.
  - This corresponds to a small perturbation of the $b$ vector.
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- This corresponds to a small perturbation of the $b$ vector.
**ShadowVertex**

Let $x_0$ be some initial basic feasible solution, and let $c_0$ be a vector for which $c_0^T x_0$ is optimal. Define:

$$\max (1 - \lambda)c_0^T x + \lambda c^T x$$

s.t. 

$$Ax = b$$

$$x \geq 0$$
**ShadowVertex**

- Let \( x_0 \) be some initial basic feasible solution, and let \( c_0 \) be a vector for which \( c_0^T x_0 \) is optimal. Define:

\[
\max (1 - \lambda) c_0^T x + \lambda c^T x \\
\text{s.t. } A x = b \\
\quad x \geq 0
\]

- Maintain an optimal solution for \( \lambda \) going from 0 to 1.
Pivoting rules

**ShadowVertex**

- Let $x_0$ be some initial basic feasible solution, and let $c_0$ be a vector for which $c_0^T x_0$ is optimal. Define:

  $$
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  \text{s.t. } A x = b \\
  x \geq 0
  $$

- Maintain an optimal solution for $\lambda$ going from 0 to 1.
- This corresponds to moving along edges of a 2-dimensional projection of the polytope (a “shadow”).
Shadows Vertex

Let $x_0$ be some initial basic feasible solution, and let $c_0$ be a vector for which $c_0^T x_0$ is optimal. Define:

$$\max \ (1 - \lambda) c_0^T x + \lambda c^T x$$

s.t. $Ax = b$

$x \geq 0$

Maintain an optimal solution for $\lambda$ going from 0 to 1.

This corresponds to moving along edges of a 2-dimensional projection of the polytope (a “shadow”).

Vertices and edges of the projection correspond to vertices and edges of the original polytope.
Pivoting rules

**ShadowVertex**

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- Maintain an optimal solution for $\lambda$ going from 0 to 1.
- This corresponds to moving along edges of a 2-dimensional projection of the polytope (a “shadow”).
- Vertices and edges of the projection correspond to vertices and edges of the original polytope.
- Spielman and Teng (2004) gave a *smoothed analysis* of the **ShadowVertex** pivoting rule, showing that it is polynomial under certain perturbations of the linear program.
All the previous pivoting rules are known to require exponentially many steps in the worst case:

- **LargestCoefficient**: Klee and Minty (1972), the Klee-Minty cube\(^1\).
- **LargestIncrease**: Jeroslow (1973).
- **SteepestEdge**: Goldfarb and Sit (1979).
- **Bland’s rule**: Avis and Chvátal (1978).
- Amenta and Ziegler (1996) gave a unified view of all these lower bounds.

\(^1\)Picture from Gärtner, Henk and Ziegler (1998)
More pivoting rules

- **RandomEdge**
  - Let a uniformly random non-basic variable with positive coefficient enter the basis.
More pivoting rules

- **RandomEdge**
  - Let a uniformly random non-basic variable with positive coefficient enter the basis.

  - Pick a uniformly random facet that contains the current vertex, and recursively find an optimal solution within that facet. If possible, make an improving pivot leaving the facet and repeat.

This randomized pivoting rule finds an optimal solution in an expected subexponential, $2^{O(\sqrt{n-d} \log n)}$, number of steps.

- **Randomized Bland's rule**
  - Reorder the indices of the variables according to a random permutation and use Bland's rule.

- **LeastEntered**, Zadeh (1980)
  - Pick the non-basic variable that has previously entered the basis the fewest number of times.
More pivoting rules

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More pivoting rules

- **RandomEdge**
  - Let a uniformly random non-basic variable with positive coefficient enter the basis.

- **RandomFacet, Kalai (1992) and Matoušek, Sharir and Welzl (1992)**
  - Pick a uniformly random facet that contains the current vertex, and recursively find an optimal solution within that facet. If possible, make an improving pivot leaving the facet and repeat.
  - This randomized pivoting rule finds an optimal solution in an expected subexponential, $2^{O(\sqrt{(n-d)\log n})}$, number of steps.

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  - Reorder the indices of the variables according to a random permutation and use Bland’s rule.

- **LeastEntered, Zadeh (1980)**
  - Pick the non-basic variable that has previously entered the basis the fewest number of times.
Friedmann, Hansen and Zwick (2011) proved lower bounds of subexponential form \(2^{\Omega(d^\alpha)}\), for \(\alpha < 1\) for the worst-case expected number of steps of the pivoting rules:
- \textsc{RandomEdge}
- \textsc{RandomFacet}
- \textsc{Randomized Bland’s rule}

Friedmann (2011) proved a subexponential lower bound for the worst-case number of steps required for the \textsc{LeastEntered} pivoting rule.

These lower bounds are based on a tight connection between \textbf{Markov decision processes} and linear programs.
Solving linear programs

- Linear programs can be solved in polynomial time:
  - Khachiyan (1979): The ellipsoid method
  - Karmarkar (1984): The interior point method
  - Best complexity results - Renegar (1988), Gonzaga (1989), Roos and Vial (1990): $O(n^3 L)$ arithmetic operations, where $L$ is the bit complexity. Based on the interior point method.
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- Finding a (strongly) polynomial bound for $T(d, n)$ and $T_R(d, n)$ is a major open problem in linear programming.
  - A polynomially bounded pivoting rule that performs each step in polynomial time would give such a bound.
The **RandomFacet** pivoting rule

  1. Pick a uniformly random facet $f$ that contains the current basic feasible solution $x$.
  2. Recursively find the optimal solution $x'$ within the picked facet $f$.
  3. If possible, make an improving pivot from $x'$, leaving the facet $f$, and repeat from (1). Otherwise return $x'$. 

A dual variant of the **RandomFacet** pivoting rule was discovered independently by Matoušek, Sharir and Welzl (1992).
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\[ P \]

\[ f_1 \]

\[ f_2 \]

\[ f_3 \]

\[ x \]

\[ x' \]
If possible, make an improving pivot from $x'$, leaving the facet $f_i$, and repeat from the beginning. Otherwise return $x'$. 
Note that if the facets $f_1, \ldots, f_d$ containing $x$ are ordered according to their optimal value, then from $x''$ we never visit $f_1, \ldots, f_i$ again.
The RandomFacet pivoting rule

- The number of pivoting steps for a linear program with $d$ variables and $n$ constraints, including non-negativity constraints, is at most:

$$f(d, n) \leq f(d - 1, n - 1) + 1 + \frac{1}{d} \sum_{i=1}^{d} f(d, n - i)$$

with $f(d, n) = 0$ for $n \leq d$. 
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  with $f(d, n) = 0$ for $n \leq d$.

- Solving the corresponding recurrence gives:

  \[ f(d, n) \leq 2^{O(\sqrt{(n-d) \log n})} \]
The **RandomFacet** pivoting rule can also be applied to the dual LP, which has \( n - d \) free variables and \( n \) inequality constraints. It follows that:

\[
T_R(d, n) = \min \left\{ 2O(\sqrt{(n-d) \log n}), 2O(\sqrt{d \log n}) \right\}
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- Clarkson (1988) showed that:

$$T_R(d, n) = O(d^2 n + d^4 \sqrt{n \log n} + T_R(d, 9d^2) d^2 \log n)$$
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$$T_R(d, n) = O(d^2 n + 2^{O(\sqrt{d \log d})})$$

This is the best known bound for $T_R(d, n)$. I.e., the best bound independent of the bit complexity.
RandomFacet: Non-recursive version

The recursion of the RandomFacet pivoting rule can be unrolled, and the algorithm can be equivalently stated as:

1. Start with a random permutation $x_1, \ldots, x_d$ of the non-basic variables.

2. Let $x_i$ be the first variable with positive coefficient according to the permutation. Make a pivot exchanging $x_i$ with some other variable $x$ in the basis.

3. Replace $x_i$ by $x$ in the list of non-basic variables and randomly permute the first $i$ variables. Repeat from (2).

The procedure resembles the Randomized Bland’s rule, but the expected number of steps is different.

Open problem: Is there a subexponential upper bound on the expected number of pivoting steps performed by the Randomized Bland’s rule?
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Basic feasible solutions give immediate lower bounds on the optimal value $z^*$. Is there a simple way to get upper bounds?
Duality

\[
\text{maximize} \quad c^T x \\
\text{s.t.} \quad Ax = b \\
\quad x \geq 0
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- If we can construct a linear combination of the equality constraints \( y^T (Ax) = y^T b \), for \( y \in \mathbb{R}^n \), such that \( c^T x \leq y^T (Ax) \), then \( y^T (Ax) = y^T b \) is an upper bound on \( z^* \).
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The problem of finding the best such upper bound can be formulated as a dual linear program $(D)$. The original linear program $(P)$ is referred to as primal.
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\[
(P) \quad \text{maximize} \quad c^T x \\
\text{s.t.} \quad Ax = b, \quad x \geq 0
\]

\[
(D) \quad \text{minimize} \quad b^T y \\
\text{s.t.} \quad A^T y \geq c
\]

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- The problem of finding the best such upper bound can be formulated as a **dual** linear program \((D)\). The original linear program \((P)\) is referred to as **primal**.
- By the **strong duality** theorem, \((P)\) and \((D)\) have the same value, assuming that \((P)\) is feasible and has a maximal value.
Consider a primal linear program \((P)\) and its dual \((D)\).
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Recall that the non-basic variables for some basis \(B\) correspond to facets that contain the basic feasible solution \(x_B\).
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I.e., \(x_i\) is fixed to 0, which is like removing the \(i\)'th column of \(A\), or for the dual like removing the \(i\)'th constraint.
Let $H$ be the set of constraints (halfspaces) of the dual. Every subset of constraints $G \subseteq H$ defines a linear program.

If $z_G$ is finite, then the corresponding primal LP has the same optimal solution.

A basis for the dual refers to $n$ linearly independent constraints. I.e., a basic solution.

A constraint $h$ is violated by $B$ if $z_B < z_{B \cup \{h\}}$.

If $h$ is violated by a basis $B$, then $z_B$ is not optimal for the corresponding primal LP, and adding $h$ to the basis must be an improving pivot.
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The dual \textsc{RandomFacet} algorithm starts with a basis $B$ such that $z_B > -\infty$. 
The dual RANDOMFACET algorithm starts with a basis $B$ such that $z_B > -\infty$.

Any basic feasible solution of the primal LP gives such a basis, but the algorithm works even if the corresponding basic solution in the primal is not feasible.
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RANDOMFACET, dual view, Matoušek, Sharir and Welzl (1992):

1. Remove a uniformly random constraint $h \in H$ that is not in the current basis $B$.
2. Recursively find an optimal basis $B'$ for $H \setminus \{h\}$.
3. If $B'$ violates $h$ repeat from the beginning, starting with the optimal basis $B''$ for $B' \cup \{h\}$. Otherwise return $B'$. 
Order constraints $h \in H \setminus B$ such that:

$$z_{H \setminus \{h_1\}} \leq z_{H \setminus \{h_2\}} \leq \cdots \leq z_{H \setminus \{h_i\}} \leq \cdots \leq z_{H \setminus \{h_{m-n}\}}$$
Repeating the analysis

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- The number of steps is bounded by:

$$f_D(k, m) = f_D(k, m-1) + 1 + \frac{1}{m-n} \sum_{i=1}^{m-n} f_D(k - i, m)$$

where $k$ is the number of unfixed constraints (the "hidden dimension"), and $f_D(m, k) = 0$ for $m \leq k$ or $k \leq 0$.

- Again, $f_D(k, m) \leq 2^{O(\sqrt{k \log m})}$. 
An **LP-type problem**, \((H, \omega)\), is defined as follows:

- \(H = \{1, \ldots, m\}\) is a finite set.
- \(\omega : 2^H \rightarrow \mathcal{W}\) is a function that maps subsets of \(H\) to a linearly ordered set \((\mathcal{W}, \leq)\) with minimal value \(-\infty\), such that:
  1. **Monotonicity**: For all \(F \subseteq G \subseteq H\), \(\omega(F) \leq \omega(G)\).
  2. **Locality**: For all \(F \subseteq G \subseteq H\) with \(-\infty < \omega(F) = \omega(G)\), and any \(h \in H\):

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\omega(G) < \omega(G \cup \{h\}) \implies \omega(F) < \omega(F \cup \{h\})
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- **Goal:** Find an optimal basis for $H$. 

The dual RandomFacet algorithm can be applied to any LP-type problem.
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Overview

- **Lecture 1:**
  - Introduction to linear programming and the simplex algorithm.
  - Pivoting rules.
  - The `RANDOMFACET` pivoting rule.

- **Lecture 2:**
  - The Hirsch conjecture.
  - Introduction to Markov decision processes (MDPs).
  - Upper bound for the `LARGESTCOEFFICIENT` pivoting rule for MDPs.

- **Lecture 3:**
  - Lower bounds for pivoting rules utilizing MDPs. Example: `BLAND’S RULE`.
  - Lower bound for the `RANDOMEDGE` pivoting rule.
  - Abstractions and related problems.