Alliances and Bisection Width for Planar Graphs

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Abstract. An alliance in a graph is a set of vertices (allies) such that each vertex in the alliance has at least as many allies (counting the vertex itself) as non-allies in its neighborhood of the graph. We show that any planar graph with minimum degree at least 4 can be split into two alliances in polynomial time. We base this on a proof of an upper bound of n on the bisection width for 4-connected planar graphs with an odd number of vertices. This improves a recently published $n + 1$ upper bound on the bisection width of planar graphs without separating triangles and supports the folklore conjecture that a general upper bound of n exists for the bisection width of planar graphs.

1 Introduction

An *alliance* is a set of vertices (allies) such that any vertex in the alliance has at least as many allies (including the vertex itself) as non-allies in its neighborhood of the graph. The alliance is said to be *strong* if this holds even without including the vertex itself among the allies. Alliances of vertices in graphs were introduced by Kristiansen et al. [11] to model among other things alliances of individuals or nations but appear many places in the literature under different names: Flake et al. [8] refer to a strong alliance as a *community* and base their work on the assumption that web pages related to each other form communities in the web graph. Gerber and Kobler [10] look at what they refer to as the *Satisfactory Graph Partition Problem* where the objective is to partition a graph into two strong alliances. A partition of a graph into strong alliances can also be viewed as a so called *Nash stable partition* of an *Additive Hedonic Game* [13]. As mentioned above, alliances have been used to model scenarios that might be planar of nature, so in this paper we focus on the problem of partitioning a *planar* graph into two alliances. In Section 2 we show how to compute such a partition in polynomial time for any planar graph with minimum degree at least 4. To prove this, we need an upper bound of n on the bisection width of 4-connected planar graphs with an odd number of vertices. We prove this upper bound in Section 3.

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This tight upper bound is an improvement over the recently published [7] $n+1$ upper bound for planar graphs without separating triangles, and it supports the folklore conjecture [7], that a general upper bound of n exists for the bisection width of planar graphs.

1.1 Preliminaries

Consider the connected graph G with vertex set V and edge set E where $|V| = n$ and $|E| = m$. The degree $d(v)$ of a vertex v in G is the number of edges incident to v in G. Similarly, for a subset $X \subseteq V$ we define the degree $d_X(v)$ of a vertex v in the subgraph of G induced by $X \cup \{v\}$ as $d_X(v) = |\{u \in X : \{v,u\} \in E\}|$. We denote the minimum degree of the vertices in G as δ. A graph G is *k-connected* when at least k vertices are required to be removed in order to disconnect G. A *clique* is a fully connected graph and a *maximal planar graph* is a planar graph with the property that the addition of any new edge destroys planarity. An *alliance* in G is a non empty set $A \subseteq V$ such that $\forall u \in A : d_A(u) + 1 \geq d_{V-A}(u)$. Throughout this paper when considering a planar graph, we will implicitly consider an embedding of the graph. A *separating triangle* in a planar graph is a triangle where both the interior and the exterior are non-empty. This definition can be tightened giving the notion of a *strong alliance* which is a non empty set $A \subseteq V$ such that $\forall u \in A : d_A(u) \geq d_{V-A}(u)$. A *partition* of G is a collection of non-empty disjoint subsets $V_1 \dots V_l$ of V such that $\bigcup_{i=1}^{l} V_i = V$. For a partition of G into two subsets V_1 and V_2 we will denote the set of edges crossing this partition as $e(V_1, V_2) = \{\{u,v\} \in E : u \in V_1 \wedge v \in V_2\}$. A *bisection* of G is a partition of G into V_1 and V_2 such that $||V_1| - |V_2|| \leq 1$ and the *bisection width* of G is defined as the minimum $|e(V_1, V_2)|$ over all bisections.

1.2 Related Work

The problem of partitioning a graph into two strong alliances is NP-hard if we put no restrictions on the graph [2]. There are however classes of graphs for which we can decide whether a partition into strong alliances exists and compute it in polynomial time. Examples of such classes are graphs with maximum degree at most 4 and graphs with girth at least 5 and minimum degree at least 3 [2, 3].

For a general graph G, the computational complexity of partitioning G into two alliances is an open problem [4]. Fricke et al. [9] show that any graph G contains an efficiently computable alliance with no more than $\lceil \frac{n}{2} \rceil$ vertices, while the problem of deciding whether an alliance with less than k members exists in G is NP-complete if k is part of the input. This even holds if G is planar [6].

Fan et al. [7] prove an upper bound of $n+1$ for the bisection width for planar graphs without a separating triangle and an upper bound on $n-2$ for the bisection width for any triangle-free planar graph. The latter upper bound has subsequently been strengthened by Li et al. [12].

2 Alliances in Planar Graphs

In this section we show that for planar graphs with minimum degree at least 4 there exists a partition of the vertices into two alliances and that this partition can be computed in polynomial time. This is trivially also true for planar graphs with minimum degree 1 (let one alliance consist of a single vertex with degree 1), while for planar graphs with minimum degree 2 and 3 it is easy to find examples which show that not all such graphs can be partitioned into alliances. See Figure 1.

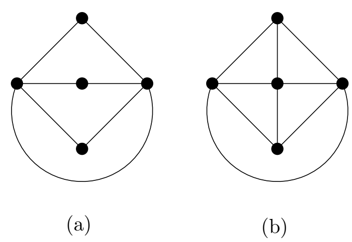

Fig. 1: Examples of planar graphs that can not be partitioned into alliances.

In Lemma 1 we characterize a group of general graph partitions that can be refined into an alliance partition using a polynomial time algorithm. In the proof of Theorem 1 we then show how to construct such a partition for planar graphs with minimum degree 4 in polynomial time. Lemma 1 is a precise formulation of the well known principle [4, 9] that a partition into two sets of vertices forms a good starting point for obtaining a partition into alliances if the number of crossing edges is relatively small compared to the cardinality of the smallest set of vertices.

Lemma 1. *A graph G can be partitioned into two alliances if there exists a partition V_1, V_2 of G such that*

$$|e(V_1, V_2)| - 2\min(|V_1|, |V_2|) < \delta - 2 \ . \tag{1}$$

The alliances can be computed in polynomial time if V_1 and V_2 can be obtained in polynomial time.

Proof. Let V_1, V_2 be a partition of G satisfying (1). We now run the following simple algorithm:

1. Let $A_1 = V_1$ and $A_2 = V_2$.
2. If A_1 and A_2 both are alliances or if one of them is empty we stop. Otherwise we go to step 3.
3. Assume that A_1 is not an alliance (otherwise we process A_2 similarly). There must be a $u \in A_1$ with $d_{A_1}(u) + 1 < d_{A_2}(u)$. We now move u from A_1 to A_2 and go to step 2.

The number of crossing edges $|e(A_1, A_2)|$ decreases with 2 or more every time step 3 is executed so the algorithm must stop after no more than $\frac{m}{2}$ steps. Assume that the algorithm stops because A_1 is empty and let u be the last vertex to leave A_1. We now consider the point in time where $A_1 = \{u\}$:

$$d_V(u) = |e(A_1, A_2)| \leq |e(V_1, V_2)| - 2(\min(|V_1|, |V_2|) - 1) \ .$$

We obtain a contradiction since (1) implies that the right hand side is less than δ. We conclude that the algorithm can not stop emptying A_1 or A_2. It has to stop with A_1 and A_2 being alliances. □

Theorem 1. *Any planar graph with $\delta \geq 4$ can be partitioned into two alliances in polynomial time.*

Proof. We start by expanding the graph by adding edges until it is a maximal planar graph which can be done in polynomial time. We now consider two cases:

The expanded graph has a separating triangle: A separating triangle has vertices both inside and outside of the triangle. Let V_1 be the vertices on the side of the triangle containing the fewest vertices and let $V_2 = V \setminus V_1$. There can be no more than one vertex in V_1 having edges to all three vertices in the separating triangle so $|e(V_1, V_2)| \leq 2|V_1| + 1$. This inequality also holds in the original graph so we can now use Lemma 1. The detection and processing of the separating triangle case is easily done in polynomial time.

The expanded graph does not have a separating triangle: In this case the graph is 4-connected since all maximal planar graphs without a separating triangle are 4-connected [5] and thus contains a hamiltonian cycle computable in linear time [1]. Fan et al. [7] show how to efficiently compute a bisection V_1, V_2 of V with $|e(V_1, V_2)| \leq n + 1$ for such a graph. This makes it possible for us to apply Lemma 1 in the case where n *is even* but for n *odd* an upper bound on n for the bisection width is needed to make inequality (1) hold. In Section 3 we prove Theorem 2 stating the existence of an efficiently computable bisection V_1, V_2 with $|e(V_1, V_2)| \leq n$ for any 4-connected planar graph $G(V, E)$ with an odd number of vertices. We now use Lemma 1 in the case where n is odd. □

As mentioned above, Fricke et al. [9] have shown that any graph contains an alliance with no more than $\left\lceil \frac{n}{2} \right\rceil$ members. We can now improve this upper bound for planar graphs with $\delta \geq 4$:

Corollary 1. *Any planar graph with $\delta \geq 4$ contains an alliance with no more than $\left\lfloor \frac{n}{2} \right\rfloor$ members.*

3 An Upper Bound for the Bisection Width

We now show that a bisection V_1, V_2 with $|e(V_1, V_2)| \leq n$ can be computed in polynomial time for any 4-connected planar graph with an odd number of vertices. Some of the techniques used are similar to the techniques used by Fan et al. [7] but we also use other techniques and the analysis is considerably more complicated compared to the analysis of Fan et al.. Since the bisection width never increases when removing edges from a graph, it is sufficient to only consider maximal 4-connected planar graphs with an odd number of vertices.

Lemma 2. *A maximal 4-connected planar graph with an odd number of vertices has a vertex u with $d(u) \geq 5$ such that $G - u$ is Hamiltonian. The vertex u and the hamiltonian cycle of $G - u$ can be found in polynomial time.*

Proof. Consider a maximal 4-connected planar graph G with an odd number of vertices. There is at least one node u in G with $d(u) \geq 5$ since otherwise we would have $\sum_{v \in V} d(v) = 2m = 2(3n - 6) \leq 4n$ that could only happen if $n \leq 5$ which would contradict 4-connectedness. The graph G is 4-connected so the graph $G - u$ has a Hamiltonian cycle computable in polynomial time as showed by Thomas and Yu [14]. □

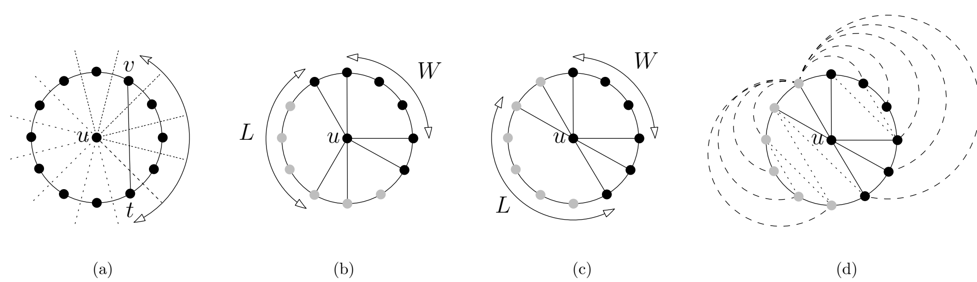

Fig. 2: Illustrations of a configuration. Figure (a) shows the hamiltonian cycle with its k hamiltonian bisections (the dotted lines) and the cycle length of edge $\{v, t\}$. Figure (b) shows a single hamiltonian bisection where the vertices are colored according to which side of the bisection they belong to. Also, it shows L and W for the configuration. Figure (c) shows a compacted neighbor configuration with L and W. Figure (d) shows a heavy compacted neighbor configuration where the dotted edges are the inner edges of the configuration and the dashed edges are the outer edges of the configuration.

Let G be a maximal 4-connected graph with an odd number of vertices and let u be a vertex in G with $d(u) \geq 5$ and C a Hamiltonian cycle in $G - u$. We will say that the tuple (G, u, C) represents a *configuration* of G. For any such configuration, there are essentially $k = \left\lfloor \frac{n}{2} \right\rfloor$ different ways to split C into two connected and equally sized parts. From these parts, we construct a *hamiltonian bisection* V_1, V_2 of G by adding u to the part where it has the most neighbors i.e. the part that minimizes $|e(V_1, V_2)|$ (ties are broken arbitrarily). Refer to Figure 2(a) and 2(b). In the following we let $T(G, u, C)$ denote the sum of $|e(V_1, V_2)|$ over the k possible hamiltonian bisections of (G, u, C). We will sometimes omit the arguments if they are clear from the context. The *cycle length* of an *edge* $\{v, t\}$ in $G - u$ is the minimum distance between v and t in the graph induced by the cycle. The contribution to $T(G, u, C)$ of an edge in $G - u$ is precisely the cycle length of the edge. Refer to Figure 2(a). We let L denote the length of the longest path along C starting and ending at a neighbor from u but visiting no other neighbors of u and let W denote the length of the second longest such path. Refer to Figure 2(b).

We refer to the configuration (G, u, C) as a *compacted neighbor configuration* if the neighbors of u can be divided into two subsets N_1 and N_2 of size $\left\lfloor \frac{d(u)}{2} \right\rfloor$ and $\left\lceil \frac{d(u)}{2} \right\rceil$ respectively such that each subset occupies a connected subpath of the hamiltonian cycle C. Refer to Figure 2(c). The *inner edges* are the edges on the same side of C as u. The inner edges that are not incident to u are naturally grouped into (at most) two groups in a compacted neighbor configuration. A compacted neighbor configuration is called *heavy* if the edges from both these groups have cycle lengths $2, 3, 4, \ldots, k, k-1, k-2, k-3, \ldots$ (for both groups we start the sequence from the left) and if the set of *outer edges* has two edges of length 2, two edges of length 3, ... , two edges of length $k-1$ and one edge of length k. Refer to Figure 2(d).

In what follows, we will show that $T(G, u, C) < k(n+1)$ for any configuration (G, u, C) of a maximal 4-connected planar graph with an odd number of vertices. Since $T(G, u, C)$ is the sum of bisection sizes for the k hamiltonian bisections this implies that there exists at least one hamiltonian bisection V_1, V_2 such that $|e(V_1, V_2)| \leq n$ which then gives us the upper bound on the bisection width. To prove $T(G, u, C) < k(n + 1)$ we will first show that the heavy compacted neighbor configurations can be considered as a set of worst case configurations such that for any configuration (G, u, C) there exists a heavy compacted neighbor configuration (G', u', C') where $T(G, u, C) \leq T(G', u', C')$. We then exploit that the heavy compacted neighbor configurations are reasonably simple such that $T(G', u', C') < k(n + 1)$ can be shown for this set of configurations.

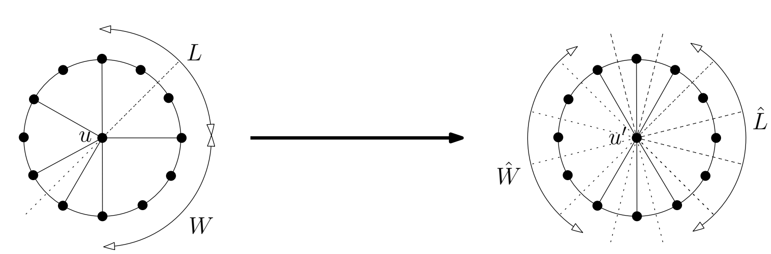

Fig. 3: In the configuration (G, u, C) to the left, the hamiltonian bisections where edges incident to u contribute with $\lfloor d(u)/2 \rfloor$ are shown with dotted lines. Similarly, in the configuration $(\hat{G}, u', C)$ to the right, the hamitonian bisections where edges incident to u' contribute with $\lfloor d(u')/2 \rfloor$ are shown with dotted lines.

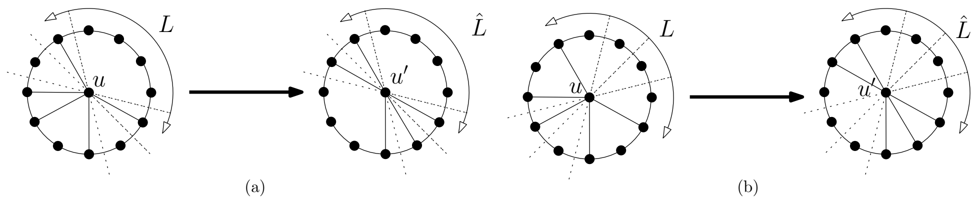

Fig. 4: In (a) we show those hamiltonian bisections which fully contain the vertices on the cycle path corresponding to L in (G, u, C) (to the left) and in $(\hat{G}, u', C)$ (to the right). In (b) we show those hamiltonian bisections which does not fully contain the vertices in (G, u, C) (to the left) and in $(\hat{G}, u', C)$ (to the right).

Lemma 3. *If (G, u, C) is a configuration then it is possible to construct a heavy compacted neighbor configuration (G', u', C) where G and G' have the same number of vertices and $d(u) = d(u')$ such that $T(G, u, C) \leq T(G', u', C)$.*

Proof. Let (G, u, C) represent an arbitrary configuration. We now remove those edges in G that are not on C and not incident to u. We then replace u (and the edges incident to u) with a vertex u' with $d(u) = d(u')$ such that the resulting configuration $(\hat{G}, u', C)$ is a compacted neighbor configuration. Finally, we put in edges to create the graph G' such that (G', u', C) is a heavy compacted neighbor configuration. Below, we first argue that the contribution to T of edges incident to u in G is not higher than the contribution to T of edges incident to u' in G'. Secondly, we argue that the contribution to T of edges in $G - u$ is not higher than the contribution to T of edges in $G' - u'$.

Edges incident to u': We separate our analysis into a case analysis based on the value of L in G. The values of L and W in $\hat{G}$ are denoted by $\hat{L}$ and $\hat{W}$ respectively.

Case 1: $L \leq k$: We consider the following subcases:

- If $2L + d(u) - 2 \leq 2k$ we build the compacted neighbor configuration $(\hat{G}, u', C)$ such that $\hat{L} - \hat{W}$ is minimized (0 or 1). Refer to Figure 3. The contribution to T of edges incident to u' is $k \left\lfloor \frac{d(u)}{2} \right\rfloor$ which is the maximum obtainable value since u' always chooses to join the partition which contributes the least to T. Refer to Figure 3. The condition $2L + d(u) - 2 \leq 2k$ makes it possible for us to obtain the $k \left\lfloor \frac{d(u)}{2} \right\rfloor$ contribution to T from edges incident to u' and at the same time obtain $\hat{L} \geq L$ and $\hat{W} \geq W$ that is important when we consider the contribution from the other edges.
- If $2L + d(u) - 2 > 2k$ we build the compacted neighbor configuration $(\hat{G}, u', C)$ such that $L = \hat{L}$ and such that the nodes forming the long paths along C with no neighbors of u of u' respectively are the same. Refer to Figure 4. For each of the k hamiltonian bisections in $(\hat{G}, u', C)$

we now show that the number of crossing edges incident to u' has not decreased compared to the corresponding (same partition of C) hamiltonian bisection in (G, u, C).

- We first consider a bisection V_1, V_2 where the vertices on the path along C of length $L = \hat{L}$ is fully contained within either V_1 or V_2 – say V_1. In this case, u' must choose to join V_2. The number of neighbors of u' in V_1 is at least as high as the number of neighbors of u in V_1 in G so the number of crossing edges for such a bisection has not decreased. Refer to Figure 4(a).
- We now consider a bisection where the vertices on the path along C of length $L = \hat{L}$ are not fully contained within either side of the bisection. When u' has chosen a side of the bisection u' has only crossing edges to members of either N_1 or N_2 (the two groups of neighbors of u'). If u' has $\left\lfloor \frac{d(u')}{2} \right\rfloor$ crossing edges the case is clear. Otherwise, the number crossing edges has not dropped since every node on the other side of the cut and not on the long path is a neighbor to u'. Refer to Figure 4(b).

Case 2: $L > k$: We build the compacted neighbor configuration $(\hat{G}, u', C)$ with $L = \hat{L}$. Consider a bisection V_1, V_2 of $(\hat{G}, u', C)$. When u' chooses side of the bisection u' can not have crossing edges to both N_1 and N_2. If there are no crossing edges the same would be the case for the corresponding bisection of the original configuration. Refer to Figure 5(a). If there are crossing edges then the number of neighbors on the other side can not have decreased. Refer to Figure 5(b)

Edges not incident to u': Since C is in both G and G' the edges on C obviously contribute with the same to T. We now consider the edges in G not incident to u and not on C and the edges of G' not incident to u' and not on C. Fan et al. [7] show how to eliminate any triangle of such edges and obtain a new set of edges with higher cycle lengths by replacing some of the edges and Fan et al. also argue that repeated elimination of triangles will produce a heavy configuration – we refer to [7] for more details. The fact that $\hat{L} \geq L$ and $\hat{W} \geq W$ makes it possible to use this technique and obtain a one-to-one correspondence between the two sets of edges considered such that any edge in the G-set is matched with an edge in the G'-set with the same cycle length or a bigger cycle length. The contribution to T of these edges can consequently not decrease during the transformation. □

Lemma 4. *Let (G, u, C) be a heavy compacted neighbor configuration with $d(u)$ even. The contribution to $T(G, u, C)$ of the edges incident to u is*

$$\frac{d(u)^2}{4} + W\frac{d(u)}{2} - \frac{d(u)}{2} \ .$$

Proof. We group the edges incident to u into pairs such that a pair of edges cuts C into two pieces with the same number of neighbors of u. For a given

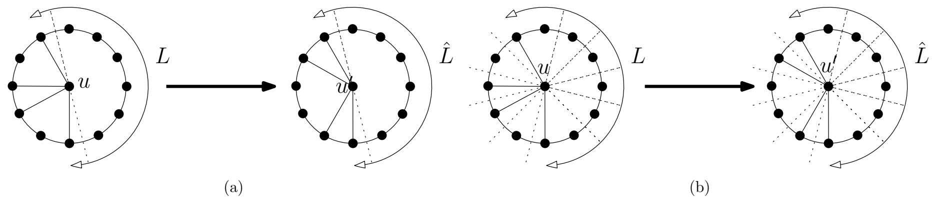

Fig. 5: In (a) we illustrate the case where there are no crossing edges for a hamiltonian bisection in G and the corresponding bisection of $\hat{G}$. In (b) we show the bisections where there are crossing edges in which case the number of crossing edges can not have decreased in $\hat{G}$.

hamiltonian bisection the contribution to $T(G,u,C)$ of a pair is 1 if the endpoints of the edges are separated and 0 otherwise. There are $W+\frac{d(u)}{2}-1$ bisections that separate each pair so it is now easy to compute the contribution to $T(G,u,C)$ of the edges incident to u:

$$\frac{d(u)}{2}(W+\frac{d(u)}{2}-1) \ .$$

□

Lemma 5. *If (G,u,C) is a heavy compacted neighbor configuration then we have the following:*

$$T(G,u,C) < k(n+1) \ .$$

Proof. We divide the proof into three cases.

Assume that $L \geq k-1$ and that $d(u)$ is even: We compute T in the following way:

$$T = 2k + \left(k + 2\sum_{i=2}^{k-1} i\right) + \left(k + 2\sum_{i=2}^{k-1} i - \sum_{i=1}^{d(u)-3}(W+i)\right) + \left(\frac{d(u)^2}{4} + W\frac{d(u)}{2} - \frac{d(u)}{2}\right) \ .$$

The first term is the sum of cycle lengths from the edges on the cycle, the second term is the sum of cycle lengths for the outer edges, the third term is the sum of cycle lengths for the inner edges not incident to u, and the fourth term is the contribution from edges incident to u given by Lemma 4. We now use $\sum_{i=2}^{k-1} i = \left(\frac{(k-1)k}{2} - 1\right)$ and $n = 2k+1$:

$$T - k(n+1) = \left(\frac{d(u)^2}{4} + W\frac{d(u)}{2} - \frac{d(u)}{2}\right) - \sum_{i=1}^{d(u)-3}(W+i) - 4 \ .$$

We now work on a part of this sum multiplied by 4 in order to exclusively have integers in the computation:

$$4\left(\left(\frac{d(u)^2}{4} + W\frac{d(u)}{2} - \frac{d(u)}{2}\right) - \sum_{i=1}^{d(u)-3}(W+i)\right)$$

$$= (d(u)-2)d(u) + 2Wd(u) - 4(d(u)-3)W - 2(d(u)-3)(d(u)-2)$$
$$= -d(u)^2 + 8d(u) - 12 + 12W - 2Wd(u) = -(d(u)-6)(d(u)-2) + 12W - 2Wd(u)$$
$$= -d(u)^2 + 8d(u) - 12 + 12W - 2Wd(u) = -(d(u)-6)(d(u)-2+2W) \ ,$$

and finally we get

$$T - k(n+1) = -\frac{(d(u)-6)(d(u)-2+2W)}{4} - 4 < 0 \ , \tag{2}$$

where we have used that the degree of u is at least 6.

Now assume that $L \leq k-2$ and that $d(u)$ is even: In this case we get

$$T = \sum_{i=2}^{L} i + \sum_{i=2}^{W} i + \frac{d(u)^2}{4} + W\frac{d(u)}{2} - \frac{d(u)}{2} + 2k + k^2 - 2$$
$$= \frac{L(L+1)}{2} - 1 + \frac{W(W+1)}{2} - 1 + \frac{d(u)^2}{4} + W\frac{d(u)}{2} - \frac{d(u)}{2} + 2k + k^2 - 2$$

implying

$$4T - 4k(2k+2) = 2L(L+1) - 8 + 2W(W+1) + d(u)^2 + 2Wd(u) - 2d(u) + 8k + 4k^2 - 8 - 4k(2k+2)$$
$$= 2L(L+1) + 2W(W+1) + d(u)(d(u) + 2W - 2) - 4k^2 - 16 \ .$$

We now use $W + L - 2 + d(u) = 2k$:

$$4T - 4k(2k+2) = 2L(L+1) + 2W(W+1) + (2k+2-W-L)(2k+W-L) - 4k^2 - 16$$
$$= 3L^2 + W^2 - 4kL + 4W + 4k - 16. \tag{3}$$

We now use $L \geq W$ in (3):

$$4T - 4k(2k+2) \leq 4L^2 + (4-4k)L + 4k - 16$$
$$= 4((L-1)(L-k+2) - 2) \ .$$

implying

$$T - k(n+1) \leq (L-1)(L-k+2) - 2 < 0 \text{ for } L \in \{1, 2, \ldots, k-2\} \ .$$

Now assume that $d(u)$ is odd: We remove the edge of u from the group with $\left\lceil \frac{d(u)}{2} \right\rceil$ edges that is closest to the path along the cycle corresponding to W. It is not hard to see that the contribution to T of the edges of u is unchanged after the removal of this edge. For $d(u) > 5$ there is consequently a heavy compacted neighbor configuration considered above with a higher value of T compared to the original graph. If $d(u) = 5$ we can use (2) with $d(u) = 4$ and W replaced by $W + 1$ if we subtract $W + 1$ (when the edge of u is removed as described above we *insert* an edge with cycle length $W + 1$ and obtain a heavy compacted neighbor configuration with $d(u) = 4$):

$$T - k(n+1) = -\frac{(4-6)(4-2+2(W+1))}{4} - 4 - (W+1) = -3 < 0 \ .$$

□

Theorem 2. *A bisection V_1, V_2 exists with $|e(V_1, V_2)| \leq n$ for any 4-connected planar graph $G(V, E)$ with an odd number of vertices and such a bisection can be obtained in polynomial time.*

Proof. Let $G(V, E)$ be a 4-connected planar graph with an odd number of vertices. As noted earlier, we can assume that G is a maximal planar graph without loss of generality. Lemma 2 assures that we can efficiently obtain a configuration (G, u, c). We now examine all the k hamiltonian bisections of the configuration. By using Lemma 3 and Lemma 5 we know that at least one of the hamiltonian bisections satisfies $|e(V_1, V_2)| \leq n$. □

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